The Interval of Motion in Leibniz's *Pacidius Philalethi*

> *Theophilus*: So we shall not be able to penetrate into the nature of motion unless we are led into this labyrinth <of the composition of the continuum>?
> *Pacidius*: Certainly not, for motion itself is made out of a number of continua.
> (A VI,3,548)

Leibniz's topic in his 1676 dialogue *Pacidius Philalethi* is the nature of motion; indeed the dialogue has as its subtitle *Prima de Motu Philosophia* or "A First Philosophy of Motion." Yet the *Pacidius* study of motion is a labyrinth unto itself, and although the dialogue finishes in tones of triumph rather than in *aporia*, it is hard to discern in it what Leibniz finally intends as his first philosophy of motion—or whether in the end the *Pacidius* even offers such a first philosophy at all.

The thesis of this essay is that Leibniz does advance a first philosophy of motion in the *Pacidius* and that its leading insight concerns the structure of motion: Leibniz's solution to the paradoxes concerning motion is to be found in the idea that motion everywhere displays what he calls "non-uniformity," and for Leibniz's account this means that motion is fractal in structure. To set Leibniz's account of motion into proper focus will demand rather detailed exegetical work on a few of the main lines of argument in the *Pacidius* (§§1-4 below), and it is one purpose of this essay simply to disentangle those lines of argument from the rest of the dialogue and to identify Leibniz's solution to the problem of motion clearly enough to open up the inquiry of the *Pacidius* for close critical appraisal. Some discussion of fractal mathematics as it stands today and as it is suggested in Leibniz's texts will be in order as well (§§5-6). With the *Pacidius* theory of the structure of motion in full view, a new analysis of the nature of motion in Leibniz's 1676 metaphysics will also come to light (§7).

1. Motion, Change And Minima

The inquiry into the first principles of motion begins naturally enough with the question "What is motion?" Charinus[1] answers:

> *Ch.*: I believe motion to be change of place, and I say that there is motion in the body that changes place. (A VI,3,534)[2]

The concept of place is not explored in the *Pacidius*, but the concept of change receives close attention. Relying upon classical lines of argument, Leibniz contends that there can be no "state of change" in a thing (that is, no "momentaneous" state of chang*ing* in a thing in addition to the earlier state it changes *from* and the later one it changes *to*) but rather that any change consists in something's

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being in two opposite states at two neighboring moments (A VI,3,558), where these moments are understood to be indivisible elements or "minima" of time. This analysis of change will have implications for Leibniz's account of motion since it commits him to a certain view about what we might call the topology of change and therefore to a certain view of the topology of motion. It is worth considering how he argues for it.

Leibniz takes as a simple example of change the passage from life to death.[3] Someone who dies is at each point in time either alive or else dead, for no particular state of such a person can be a state of death if she is at that moment still alive, nor a state of being alive if she is at that moment dead. Both cases would involve contradictory states and are therefore impossible. Yet as there is no third alternative to being alive and to being dead—Leibniz explicitly asserts *tertium nullum est* (A VI,3, 535)[4]—it must be that the passage from life to death occurs without any "middle moment" or "momentary state of change" through which this change is transacted. All this seems right if we grant that life and death are indeed opposite states and that *tertium nullum est*; but the interlocutors in the next few lines make an important further claim:

> *Pa*.: Tell me, Charinus, do you think that some people are dead who used to be alive?
> *Ch*.: This is certainly the case, however we may keep on arguing.
> *Pa*.: Did life end for them at some time?
> *Ch*.: Yes.
> *Pa*.: So was there some last moment of life?
> *Ch*.: Yes, there was.
> *Pa*.: Again, Charinus, do you think that some people used to be alive, who are now dead?
> *Ch*.: This is certain, too, but it is the same case as before.
> *Pa*.: Suffice for it to be certain. So did a state of death begin for them?
> *Ch*.: Yes.
> *Pa*.: And there was some first moment or beginning of this state?
> *Ch*.: Yes.
>
> (A VI,3,536)

The important claim here is that there exist both a *last* moment of life and a *first* moment of death, and from this proceeds the analysis of the specific "act of dying" (A VI,3,535) itself as a pair of *immediately* adjacent moments in time containing opposite states, one of life and one of death. Likewise the analysis of change in general takes this form: there is no single "momentaneous" state of change that occurs when change occurs but rather what we might ordinarily take to be a momentaneous state of change in fact is an aggregate of *two* neighboring moments containing *two* opposite states (A VI,3,541).

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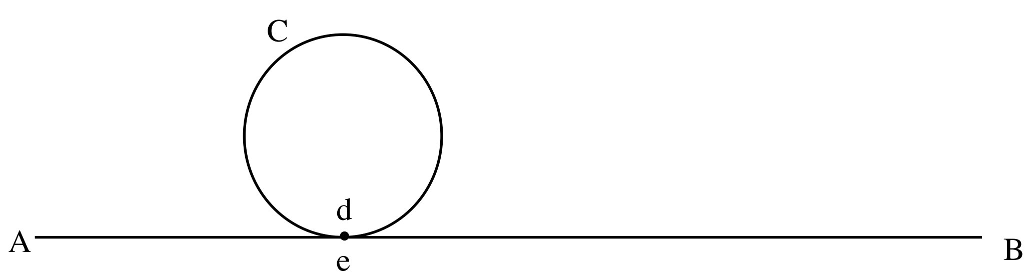
But we should note that just from life and death being opposite states and from the principle *tertium nullum est* it does *not* follow that in the act of dying a final moment of life is succeeded immediately by a first moment of death. All that follows is that either some final moment of life is succeeded only by moments of death *or* some first moment of death is *pre*ceded only by moments of being alive. (And even this weaker consequence presupposes that there be no "gaps" or "missing moments" in the series of moments in time—i.e. that the series cannot lack *both* a last moment of life *and* a first moment of death.) The proposition that some last moment of life will be immediately followed by some first moment of death is consistent with Leibniz's analysis but not yet required by it. For his argument excluding middle moments equally allows that there may simply be no last moment of life or, alternatively, no first moment of death: the series of moments may instead be densely ordered[5] so that if we suppose, for example, that there is a last moment of life, then between that last moment of life and any moment of death you like there always lies some *earlier* moment of death and so there will be no *earliest* moment of death. (Or *vice versa* for the alternative supposition of a first moment of death.)

Leibniz does not quite assert that we are forced to hold that the act of dying involves two immediately neighboring moments containing opposite states. Rather we find in the dialogue a discussion of a certain *possibility*:

> *Pa.*: Therefore it is possible for two moments, one of living and the other of non-living, to follow one immediately after the other?



> *Ch.*: Why not, when this is also possible for two points? I find this comes to mind most conveniently when I can set the thing before my eyes in some fashion. So let a perfectly rounded sphere *C* be placed on a perfectly flat table *AB*. It is clear that the sphere does not cohere with the plane, and that they have no extrema in common, otherwise one would not be able to move without the other. On the other hand, it is clear that there is no contact unless it is at a point, and there is some extremum ôr [*sive*] point *d* in the sphere that is no distance from the extremum ôr point *e* of the table. So the two points, *d* and *e*, are together, although they are not one point. (A VI,3,537)

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So it seems that we have two similar cases: in change two moments in time might follow one immediately after the other, and in contact two points might fall one immediately *next* to the other. The tone of the last passage suggests that Leibniz takes the example of contact to be the actual truth about contact—indeed, that appears to be the very point of the analogy—and so the implication here is that the possibility of two neighboring moments existing in the case of change is the actual truth about change, too.

I take Leibniz's remarks in this connection to concern the topology of change and contact. The ideas being expressed are classical in origin, descending from Aristotle's celebrated discussion of "the continuous" and "the contiguous" at *Physics* V.3,227a10-b2.[6] As Leibniz has the character Theophilus note:

> I recall that Aristotle, too, distinguishes the contiguous from the continuous in such a way that those things are *continuous* whose extrema are one, and *contiguous* whose extrema are together [*simul*]. (A VI,3,537)

This concept of "the contiguous" is not unproblematic in its application by Leibniz to the elements of change and contact. On the view of contact that results, things that are touching will have distinct boundaries—faces or edges or points on their surfaces—pressed absolutely flush against one another so that they are, as Leibniz says, "indistant" [*indistantia*] (cf. A VI,3,557). This is at once plausible as an account of our naïve or intuitive picture of contact by "touching" and yet incompatible with the (contemporary) idea that objects are embedded in a space that is a mathematically continuous manifold of points. For on the standard topological account for such a space, any two points will always be separated by a distance of some finite measure, and falling between them there will always be an intervening point—indeed, an intervening infinity of points. On this picture of the topology of space, even if the sphere and the table were mathematically precise objects with perfectly smooth outer boundaries, they could not touch in the way that Leibniz suggests that they do. There simply are no two immediately neighboring places, distinct but indistant *loci,* available in the world for the surface points *e* and *d* of the table and the sphere to be.

Translating this worry across into the discussion of change, it seems that there could be no two moments immediately next to each other in time for a pair of opposite states—the last state of life and the first state of death—to occupy. And hence there could not be any aggregate of two states immediately adjacent in time to be the *analysans* that Leibniz claims for his analysis of the nature of change.

The difficulty here is a serious one, and I know of no mathematically coherent way to spell out Leibniz's idea of distinct but indistant boundaries residing at the locus of contact or change. Of course it may be possible to introduce such points of contact or change as topological primitives an

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d go on to reconstruct some broader topology for space and time; perhaps such a stratagem could throw some light on the consequences of admitting distinct but indistant boundaries. But I doubt whether it would help much to explicate Leibniz's own thinking about contact and change, and whether the resulting topology would capture our "deep" intuitions concerning space and time. Still, the problem of reconciling the standard point-set topology for space and time with intuitive conceptions of contact and change is not unique to Leibniz, and I do not mean to rule out in advance the possibility that some coherent account of a Leibnizian topology for change and contact can be articulated.

What matters for us at this point is simply to see that in taking a certain view of change to be *possible*—not to mention actual—Leibniz already commits himself to certain kinds of claims about the topological structure of reality. Space and time are not going to be everywhere continuous in the mathematical sense that requires that any two points whatsoever will be separated by a further point lying between them. At least some two moments in time or some two points in space are going to be assigned that lie perfectly side by side, with no further times or places falling between them. And indeed every single occurrence of change or contact should give rise to such pairs of neighboring times or places, so we should expect to find them scattered throughout Leibnizian reality.

A further element of this account of change and contact that we should take care to note is its commitment to the existence of *points* and *moments* in space and time—the simple, indivisible elements that Leibniz calls the "minima" of space and time—and not just to their intervals. This is not an unorthodox view, of course, but it is nonetheless a substantive one for a first philosophy of motion; and in the *Pacidius* Leibniz is prepared to argue for it. The strategy is to show that change itself is transacted by the addition or subtraction of a minimal difference in the state of the thing that changes, and thus that the very possibility of change entails the existence of minima in reality. To this end Leibniz makes extraordinary use of the classical Sorites argument concerning change, unlimbering it in two distinct forms.

The first Sorites argument takes up the example of a pauper ceasing to be poor by successively gaining single pennies. Running through the reasoning in the familiar way, the interlocutors conclude that the gain or loss of a single penny—the minimal unit of wealth—must be able to make the difference between being rich and not rich or between being poor and not poor. Here is the very last bit of the Sorites, where Charinus is being argued out of his prior belief that the passage into or out of poverty or wealth cannot be transacted by the difference of a single penny:

> *Pa.*: Suppose a penny is given to a pauper. Does he cease to be poor?
> *Ch.*: No.
> *Pa.*: If another penny is given to him, does he cease to be poor then?
> *Ch.*: No more than before.
> *Pa.*: Therefore he does not cease to be poor if a third penny is given to him?

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Ch.: No.

Pa.: The same applies to any other one: for either he never ceases to be poor, or he does so by the gain of one penny. Suppose he ceases to be poor when he gets a thousandth penny, having already got nine hundred and ninety-nine; it is still one penny that removed his poverty.

Ch.: I can see the force of the argument, and I'm surprised I was deluded like this.

Pa.: Do you admit, then, that either nobody ever becomes rich or poor, or one can become so by the gain or loss of one penny?

Ch.: I am forced to admit this.

(A VI,3,539)

Charinus's "forced admission" that there is always a sharp boundary between being poor and not poor or rich and not rich constitutes a bold step by Leibniz, and no doubt it will strike many contemporary readers as a high price to pay in order to resolve the Sorites paradox. Still, I do not think that Leibniz obviously draws the wrong conclusion here. Every resolution of the Sorites that I am familiar with seems at least as problematic as the view that Leibniz recommends: either nobody ever becomes rich or poor, or one can become so by the addition or subtraction of a single penny.[7]

The key step in the argument for minima comes next, in the second Sorites, as Leibniz has the interlocutors switch from the example of someone's becoming rich or poor to a new example in which a movable point *A* approaches a distant fixed one *H*; "at a certain time," Leibniz writes, "it [the movable point *A*] will turn from not being near to being near"(A VI,3,540). As the character Pacidius describes it, this switch is intended to "transpose the argument from discrete to continuous quantity"(*ibid.*). By 'discrete quantity' Leibniz has in mind a notion of quantity that comes with a pre-assigned unit of measure assigned to it: baldness, for instance, is finally measured in single hairs, and wealth in pennies—the penny is the pre-assigned minimum of wealth, so to speak. A continuous quantity, by contrast, does not come with a pre-assigned unit of measure, but rather units of measure can be assigned as finely as one wishes. Motion itself, understood as a change of place, will thus involve a "continuous quantity," since Leibniz holds space to be continuous in this sense. The approach of the movable point *A* as it closes the distance to the fixed point *H* can equally well be measured in inches, or hundredths of an inch, or thousandths of an inch, and so on, right up to the theoretical limit. But just as in the first Sorites any extra measure of wealth beyond the minimum of wealth, the single penny, was actually superfluous in finally bringing about the change from being poor to not being poor ("it is still one penny that removed his poverty"), so too in this second Sorites any extra measure of distance beyond a minimum of distance must be superfluous in bringing about the change from not being near to being near. And the conclusion drawn at the end of the second Sorites argument is indeed exactly parallel to that of the first:

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> Therefore we have that either there is no way for something to become near properly and of its own accord, or something turns from being near to being not-near by the addition or subtraction of a minimum, so that there are minima in reality. (A VI,3,540)

So it is by this Sorites argument "transposed to continuous quantity" that Leibniz argues for the existence of minima in reality. This second Sorites argument is actually quite subtle and merits close study, but for now that study must be left to another occasion.[8]

What are the minima for the "continuous quantities" of space and time? Not the foot and the hour, nor the inch and the minute—nor, in general, any divisible interval of space or time, no matter how small. For the minima of space and time must finally be indivisible units: points and moments. (Lines and planes will also be counted as minima, presumably for being the minimal boundaries of spatial continua of two and three dimensions.) Leibniz holds that there are minima in matter as well: every surface, edge or vertex of any part of matter is a minimum (cf. A VI,3,552f., 555, 565f.). Below we shall have more to say about the ontological status of these minima—specifically, that on Leibniz's view they are merely "modes" of extended entities and not things in their own right—and some complications they present for his account. But for the moment we should pause to sum up the main points arising from Leibniz's analysis of change. (1) Change is an aggregate of two immediately neighboring moments in time containing the existence of two opposite states. (2) Motion, as change of place, is a continuous change or a change involving continuous quantity, and its occurrence requires the addition or subtraction of minima. (3) In general, there are minima existing in reality: points in space, moments in time, surfaces, edges and vertices of bodies.

2. The Labyrinth Of The Composition Of Motion

With those outlines of the nature of change in hand, the account of motion as change of place can now be further elucidated—if only to reveal how profound the difficulties are concerning the composition of motion. Motion of a body *x* will be a change of place, and this change will itself be an aggregate of two moments: in particular it is the aggregate composed of the last moment of *x*'s existence in one place and the first moment of its existence in the next place to which it is moved (cf. A VI,3,541). Simple enough. But we should want to know what larger picture of motion it is that issues from this small-scale analysis. What is it for something to move across a distance into what Leibniz calls "some remote place" [*loco aliquo dissito*: A VI,3,556], a place separated from the starting point of the motion by some intermediate spaces?

Let motion across such a space be an *interval* of motion. Framed in this way the question then is: what is the correct analysis of the structure of an interval of motion? The point of framing the question this way is to remain neutral with respect to a further philosophical question about the fun

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damental nature of motion that might be asked, namely, what it is for there *to be* an interval of motion. For Leibniz's view of the fundamental nature of motion is formed in light of his analysis of its structure rather than prior to it, and moreover the answer to this second question is embroiled in the very delicate issue of "realism" about motion in the *Pacidius*. The second question is to be addressed in the last section of this essay.

To elicit an account of the interval of motion from the small-scale analysis of motion as a change of place, it must be determined where the moving body exists at the next moment as it travels across the interval. For simplicity, suppose that the moving body *x* is a "movable point"—as in many of Leibniz's examples in the *Pacidius*—and thus that the interval of its motion in space is a line. Leibniz identifies three possibilities for the location of *x* in the following moment, yielding three very different pictures of the structure of motion. Each one of them will prove to be deeply problematic, and it is with the difficulties set as sharply as possible that Leibniz is forced to articulate an altogether new solution to the problem of motion. Let us now consider in turn the three problematic analyses of the structure of motion as they culminate in the central perplexities facing Leibniz's inquiry into motion.

1. *Totus locus*. The moving body *x* will at the next moment occupy *all* the points on the interval—it will be "in the whole place"(*in toto loco*: A VI,3,557), thereby being "simultaneously at the intermediate points and at the endpoints"(A VI,3,562). On this view motion would be a sort of *expansion* of the moving body as it fills the interval of space in a moment. Leibniz allows Charinus to suggest the *totus locus* option with the appeal "But see if it isn't absurd for the same body to be in several places at once"(A VI,3,557). But in the very next line Pacidius cuts off that angle, saying that those who admit a motion by which *x* reaches the remote place in a moment would *not* adopt that view, for "they would fall back into the above difficulties"(*ibid.*)—apparently the difficulties facing the hypothesis of "middle moments" or momentary states of change, namely that the moving body would be in opposite states by being in several places at once, a condition then described as "absurd"(*ibid.*).

2. *Locus dissitus*. The moving *x* will pass by means of "an instantaneous motion by a leap" across the interval into the "remote place" [*loco dissito*] "in such a way that it did not pass through the intermediate places"(A VI,3,556.) This case is rejected by Leibniz pretty nearly out of hand: such metaphysical "leaps" [*saltus*] into a remote place in which a moving body does not pass through all the intermediate points are said to be "impossible" (A VI,3,542). Clearly the same argument will apply *mutatis mutandis* to any motion across an interval into a remote place, so it's no use trying to make such a leap less painful to the mind by suggesting, say, that a very small intermediate space could perhaps be omitted in the motion and simply leaped over. For it was not the size of the interval that made the leap objectionable but the mere fact that some intermediate spaces had to be leaped over at all. Charinus puts the point vividly:

> I find these leaps [*saltus*] of yours very excruciating. For given that size has nothing to do with the matter, it seems to me just as absurd that some very small corpuscle should get from one end to the other of an arbitrarily small linelet without going through the intermediate points, as that I should be transferred to Rome in a moment, leaving out all the intermediate places in the same way, as if there weren't any in nature. (A VI,3,560)

So there will be no motion by a leap into a remote space across an intervening space no matter how small. In one draft of the dialogue Leibniz allows the interlocutors to consider the special case of motion by a leap across only an *infinitesimal* intervening space (cf. A VI,3,564-5). He rejects this proposal as well, but on different grounds. Once infinitesimal spaces are admitted, there will be a bottomless descending hierarchy of ever-smaller scales for infinitesimal spaces, and "no reason can be provided why some should be assumed rather than others"—no reason for assigning one scale of infinitesimals as the scale at which motion by a leap is admissible—"but nothing happens without a reason"(A VI,3,565). In at least this one case, it seems to me, Leibniz's use of a principle of sufficient reason is pretty well beyond reproach.[9] Motion by a leap into a remote space is thus ruled out in all cases. What this tells us is that the path of motion must be spatially continuous at least in the weak sense that the movable point *x* must proceed in order through all the points lying on that path, touching down for at least a moment at each one.

3. *Locus proximus*. The moving point *x* will exist at one of the "intermediate places"(cf. A VI,3,541) in the interval—in fact the "next place" [*locus proximus*: A VI,3,546f]" or the place immediately neighboring the initial one. Leibniz works here with a diagram with an initial place *A* and the immediately neighboring one *C*. By the second Sorites argument ("transposed to continuous quantity") given earlier, we can see that the interval from *A* to *C* must contain no halfway point; for if there were some third point in that interval from *A* to *C*, then the motion from *A* to *C* would be a leap after all. Thus the first interval crossed in *x*'s motion away from *A* is a sort of indivisible one, and Leibniz will describe it as "null" and as a "minimum." But we should take care in understanding the latter term in this context. Leibniz does *not* mean to say here that there is some indivisible point that constitutes the interval from *A* to *C*; rather, the claim is that the neighboring places *A* and *C* are at *no* distance from one another.

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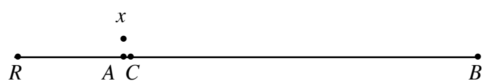
> Let there be a moving body *x* for which there are two neighboring places [*loca duo proxima*] *A* and *C* whose interval must be null ôr [*sive* = equivalently] a minimum; ôr, what is the same thing, the points *A* and *B* must be such that no point can be assumed between them, or such that if two bodies *RA* and *CB* were there, they would touch each other in the extrema *A* and *C*. So the motion is now the existences of the thing *x* in the two neighboring points *A* and *C* at the two neighboring moments. (A VI,3,546)[10]



Thus the interval from *A* to *C* is null because there is *nothing* between them—not even one point. *A* and *C* are in perfect contact or, in Leibniz's term, *indistantia*.

As emerges in a parallel discussion in the dialogue, on this *locus proximus* account the moving body passes between indistant places by virtue of being "transcreated" by God: first "vanishing" at the one place and then being "resuscitated" at the other (cf. A VI,3,559). Leibniz's doctrine of transcreation is first mentioned in the 1676 piece "On Infinite Numbers" (cf. A VI,3,500), and is understood by Leibniz to be motion "by a leap," but *not* motion by a leap in the sense that an intervening place is omitted in the passage of the body from its first location to its second, since there is no such place between immediately neighboring points. (We shall return to the subject of transcreation in connection with the question of the nature of motion at the end of this essay.)

The difficulties for the *locus proximus* proposal begin to reveal the true depth of the problems surrounding the account of the interval of motion. Leibniz has Pacidius ask, in effect, whether *x*'s motion across the interval is continuous. And by 'continuous' here, he means "the motion is not interrupted by a rest at any time, that is to say, the motion is able to last in such a way that the body [*x*][11] does not exist in any place (equal to itself)...for longer than a moment"(A VI,3,542).[12] To answer "yes," that the motion *is* continuous from start to finish, will result in a commitment to the claim that the intervals of space and time that *x* traverses are exhaustively decomposed by its motion into minimal elements—into points and moments. Here's the argument from the *Pacidius*:

> *Pa.*: If now a motion is continuous for a while, without any rest intervening through a certain space and time, then it follows that this space is composed only of points and time only of moments.
>
> *Ch.*: I would like you to show this more clearly.

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Pa.: If the present motion is an aggregate of two existences, it will be continued out of more, for we assumed it to be continuous and uniform. But different existences belong to different moments and points. And if there are nothing but further different existences immediately following each other extending through all time and place, there will, therefore, be nothing but moments and points immediately following each other in time and place.

Ch.: No doubt, for because of the *uniformity* of motion, place and time, no reason can be found for one rather than the other, since from a point a body can proceed only to the next point, and this always at the next moment following.

Pa.: So, since motion is nothing but an aggregate of diverse existences through moments and points, and is equally continuous as space and time, there are therefore points immediately following each other everywhere in space, and moments immediately following each other everywhere in time, these being the points and moments in which motion occurs by continuous succession. Therefore time will be an aggregate of nothing but moments, and space an aggregate of nothing but points.

(A VI,3,546-7)

This conclusion about space and time cannot be maintained, however, for according to Leibniz it is impossible for a continuum such as an interval of space or time to be composed out of minima or resolved into them. This is of course a staple feature of Leibniz's later remarks about the continuum, but it is in the *Pacidius* where his argument for it finds its clearest expression. And indeed the *Pacidius*'s argument against the composition of the continuum from minima has been the subject of some recent discussion in the literature, so a brief summary of it will serve here.[13]

If the continuum—say a line—is composed of minima, it must be composed of either a finite number or an infinite number of them. If a finite number, say ninety-nine minima, then the line will be divisible only into a fixed finite number of equal parts; and this contravenes what the "Geometers" have long ago demonstrated, namely that "any line whatsoever can be divided into a given number of equal parts"(A VI,3,549)—i.e. *any finite number* of equal parts. But in the present case the line will not even be divisible into 100 equal parts. And likewise for any finite number of minima you might pick to compose the continuum. If, on the other hand, the line is composed of an infinite number of minima, then (paradoxically) it can be proved that a proper part of that line also contains an infinite number of minima, for a one–one correspondence can be defined between them.[14] But then the whole will be "equal to" the part; and that contravenes the "axiom" that the whole is greater than the part (cf. A VI,3,550f.), which is absurd. Therefore the continuum cannot be composed of minima.

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There is a lesson about the nature of minima that Leibniz takes from these paradoxes—one that will prove to be both a profoundly important tenet in his metaphysics and (as we shall later see) a crucial element in the difficulties which that metaphysics encounters. Minima are not to be thought of as parts of things or things in their own right; rather, minima are only boundaries or extrema of extend things. And they only exist in reality at all by virtue of the assignment of various parts in matter or space or time.

> *Ch.*: [T]here are no points before they are assigned. If a sphere touches a plane, the locus of contact is a point; if a body is intersected by another body, or a surface by another surface, then the locus of intersection is a surface or a line, respectively. But there are no points, lines or surfaces anywhere else, and in general there are no extrema except those that are made by a dividing [*fiunt dividendo*]: nor are there any parts in the continuum before they are produced by a division [*divisione*]. (A VI,3,552-3)

Minima depend for their existence on the condition of the interval—upon how it is divided into parts—and cannot be taken to exist independently of it or to "pre-exist before an actual division"(A VI,3,564). Minima must therefore be taken to be endpoints or boundaries of intervals and not constitutive elements of them.

What we find here with Leibniz is that if motion consists in a body's existing at one moment in one place and in the following moment in an immediately neighboring place, as is proposed on this last of the three possible accounts of motion (and in fact it is a sort of variant of this *locus proximus* account that Leibniz will eventually accept), then motion itself cannot be continuous across any interval, on pain of composing the continuum from minima. If such point-by-point motion cannot be continuous, what if we instead suppose that it is not continuous but rather is interrupted by "rests" [*quiete*] during which the body remains in a single place for more than a single moment (cf. A VI,3,542). Can this tactic help? No—it proves to be precisely no help at all. For if there is some motion to be interrupted by rests, then in between the rests there must be some intervals of motion, and the question will once again arise: are these little intervals of motion continuous or interrupted by rests? Obviously it is no good to answer "this interval is not continuous, but interrupted by rests" every time, *ad infinitum*, for in the limit of this analysis we should find that the motion of body *x* across the interval is resolved *entirely* into rests and hence that *x* does not ever move after all—contrary to our hypothesis (cf. A VI,3,556, 562). So it appears that if body *x* is to move into a remote place without getting there by a leap, sooner or later there must be an interval of motion that is *not* interrupted by rests: an interval of continuous motion. And thus we are led back again to the problem of having to compose out of moments and points the continuous intervals of time and space that are traversed by *x*, which is absurd.

By now it will have begun to dawn upon us that in our inquiry into the topic of motion we have in fact been drawn into the labyrinth of the composition of the continuum; and escape seems impossible.

In the *Pacidius* Leibniz advances a solution to these difficulties that is supposed to guide us through the labyrinth. The solution consists in a certain analysis of the structure of the interval of motion. But seeing what it amounts to in the end is not as easy as we might wish it to be. In order to get as clear as possible about what that solution will be, we should take stock of the main points of the inquiry into motion that have produced the difficulties. Charinus's summary at A VI,3,562 provides the best vantage point.

> Whatever moves changes place, ôr changes with respect to place. Whatever changes is in two opposite states at two neighboring moments.
>
> If anything changes continuously, then any moment of its existence in one state is followed by a moment of existence in an opposite state. Thus in particular:
>
> If any body moves continuously, then any moment of its existence at one point of space is followed by a moment of existence at another point of space.
>
> These two points of space are either immediately next to each other, or mediately.
>
> If immediately, it follows that a line is composed of points, for the whole line will be traversed by this passage from one point to the other immediately next to it.
>
> But for a line to be composed of points is absurd.
>
> If the two points are mediately next to each other, then a body passing from one to the other in a moment will either be simultaneously at the intermediate points and at the endpoints, which is absurd; or it will make a leap, and will pass from one endpoint to the other by omitting the ones in-between. Which is also absurd.
>
> Therefore the body does not move continuously, but rests and motions are mutually interspersed.
>
> But the interspersed motion is again either continuous, or interspersed with another rest; and so on to infinity.
>
> Therefore either somewhere we will come across a pure continuous motion, which we have already shown to be absurd, or we must admit that no motion is left at all except momentaneous motion, and that everything is resolved into rests.
>
> So again we come across momentaneous motion, ôr [*seu*] a leap, which we wanted to avoid. (A VI,3,562)

Motion apparently can neither be purely continuous nor discontinuous and "by leaps," neither uninterrupted nor interspersed with rests. To pursue the consequences of any of those suppositions on

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ly leads into absurdities or else back to the same questions, and soon we are without any coherent answers.

What then are we to say about motion?

3. The Non-Uniformity Of Motion

Leibniz's solution to the difficulties of the labyrinth of the composition of motion is meant to capitalize on an assumption that was quietly made about the interval of continuous motion, namely, the idea that not only will the interval of motion be continuous, in the sense that it is uninterrupted, but also it will be *uniform*. Recall how Leibniz pointed out this assumption in a passage we saw earlier—in fact in the very passage where the problem of composing the continuum from minima was first announced:

> *Pa.*: If the present motion is an aggregate of two existences, it will be continued out of more existences, *for we assumed it to be continuous and uniform*. But different existences belong to different moments and points. And if there are nothing but further different existences immediately following each other extending through all time and place, there will, therefore, be nothing but moments and points immediately following each other in time and place.
>
> (*a little later*)
>
> *Ch.*: No doubt, for because of the *uniformity* of motion, place and time, no reason can be found for one rather than the other, since from a point a body can proceed only to the next point, and this always at the next moment following.
>
> (*with omissions*; *first italics added*; A VI,3,547)

When Leibniz says here to say that uniformity was *assumed* about the structure of motion, he is certainly right. This passage presents us with the very first occurrence of the term 'uniform' in the whole dialogue; uniformity has literally never been mentioned up to that point. But it is in this assumption of uniformity that Leibniz evidently sees the possibility of an escape from the labyrinth—an escape that will be made precisely by *denying* the uniformity of motion. We shall want to know just what it is that he means to deny here and how this denial will help to resolve the difficulties about motion.

"Uniformity of motion" in seventeenth-century physics often signifies unaccelerated motion or motion with constant velocity, i.e., the motion of a body suffering no inertial forces. And it is indeed plausible to suppose that Leibniz has unaccelerated motion in mind, especially in light of his f ollow-up remark at A VI,3,565 that the (contrary) hypothesis of *non*-uniform motion “is also consistent with reason, for there is no body which is not acted upon by those around it at every single moment.”

The picture of accelerated motion Leibniz appears to be working with at this point is quite intriguing. During the late seventeenth century the acceleration of a moving body though an interval of its motion was often dealt with mathematically as if it were due to a series of instantaneous finite impulses punctuating tiny subintervals of uniform motion so that in each successive subinterval the moving body has a fixed higher (or lower) velocity than it had in the preceding one. The graph of the accelerated motion would appear as a polygonal curve built up of little straight pieces. On this conception bodies are idealized as perfectly rigid, and the collisions among them are idealized as perfectly elastic.[15] Newton handled accelerated motion in essentially this way in the *Principia* (1687). Yet the key groundwork for this “impulse” picture was laid over thirty years earlier by Huygens in his 1656 study *De Motu corporum ex percussione* (“On the Motion of Bodies by Percussion”), parts of which were published in 1669,[16] and which I can only suppose was in 1676 already well-known to his pupil Leibniz. The impulse picture sits in contrast to the “classical contemporary” twentieth-century one in which the acceleration of a moving body through an interval occurs as the result of a *continuously* acting force that does not “spike” the motion of the body with any sudden addition of a finite quantity of force in a single instant but rather adds any finite quantity of force gradually across some finite interval of the motion. The impulse picture is not what the early modern physicists actually *believed* about acceleration, of course. Like their twentieth-century successors, they imagined accelerative forces to act continuously through an interval, but they were for the most part unable to solve the differential equations that would represent such continuously acting forces. The impulse picture makes the technical problem tractable by allowing accelerated motion to be represented as a polygonal curve—one that will only approximate its true “smooth” character—which can then be handled rather easily with the resources of algebra and Euclidean geometry. The most significant remaining steps towards the contemporary mathematical models would not be taken until the eighteenth century with the work of Euler and others. Still, the impulse picture is mathematically quite unlike the contemporary one, and it will make some difference to our understanding of Leibniz’s theory of motion to observe the distinction.

In light of that distinction, Leibniz’s remark that “there is no body which is not acted upon by those around it *at every single moment*” appears to place him in between the contemporary and the seventeenth-century pictures of accelerated motion. Moving bodies are indeed acted upon by instantaneous impulses—“instantaneous” at least in that the *onset* of the acceleration begins with a sharp addition of force at a given instant, even if no impulse actually *exists* only for a single moment (cf. A VI,4,1613)—and with each impulse the interval of motion is subdivided and the body takes on a new submotion. But the impulses do not occur merely in some handful of the moments during the f

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ull interval of motion. Rather, they rain down on the moving body at every single moment and from all directions. Since there are infinitely many distinct moments in every finite interval, the moving body will therefore suffer the impacts of infinitely many distinct forces during every interval of motion however small. The resulting motion is not merely non-uniform, it is *nowhere* uniform across *any* subinterval. But neither is it accelerated continuously in the sense of being gradually accelerated by a force that acts throughout the interval, for each little impact adds some sharp change to the motion of the body. In Leibniz's hands the impulse picture of acceleration is being "pushed into the limit" so that it resembles a continuously acting force insofar as the non-uniformity of motion is maintained throughout every subinterval however small, but yet the impulses are still "topologically" discrete, adding punctual changes in velocity and agitating the moving body slightly in a different direction at each moment.[17] (Some other passages that we shall consider below—passages occurring at A VI ,555 and 563-566—will show Leibniz articulating this picture of accelerated motion more fully.)

As intriguing as this new impulse picture of acceleration is, it is important to see that in the *Pacidius* the puzzles that Leibniz deals with are first of all *kinematic* or *phoronomic* in character rather than *dynamical*: they concern motion in the abstract and are not (primarily) addressed to the action of accelerative forces.[18] His proposal about the non-uniformity of motion is introduced on purely kinematic grounds rather than as an account of the interaction of a moving body with its neighbors. The fact that the hypothesis of non-uniformity agrees so sweetly with his dynamics is essentially a bonus—and perhaps a source of independent confirmation for the kinematic proposal.[19] Still, this harmony between Leibniz's dynamics and his kinematics allows the two to be very closely interwoven by the mathematics that underlies each, and indeed the mathematical representation of motion in the calculus naturally yields both dynamical and merely kinematic or "geometrical" readings. Let us take a moment to consider the "geometrical meaning" of the assumption that motion is uniform and what it would mean to deny it.

Uniform motion is represented mathematically as a linear function pairing positions and times. To calculate the velocity of a body undergoing uniform motion at a given instant *t*, one need only find the average velocity of an interval that includes *t*—take the difference between its positions at the endpoints of the intervals, and divide by the elapsed time—for the average velocity across the interval will be the same as the instantaneous velocity at *t*. But if the motion is *non*-uniform or accelerated, then taking the average velocity for such an interval will not in general suffice to yield the velocity at the instant *t*. Rather, one must take the average velocity of ever-narrower intervals containing *t* to get increasingly close approximations of the actual velocity at *t*. In the limit of this analysis, the velocity at *t* will be pinned down exactly. This is simply to take the first derivative of the function with respect to time *t*, equivalent to finding the slope of the tangent to the function's corresponding curve at that point. And to calculate the acceleration of the body at that moment, one takes the second derivative.

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Straightforward as this is, there are some fine points to be observed. Taking a first or second derivative of a function that describes a particle's (possibly non-uniform) motion is accomplished by applying the techniques of the differential calculus. Those techniques proceed on certain assumptions about any function, or its corresponding curve, to which they may be applied, and so there are preconditions on a function's being differentiable in the first place. In particular, the calculus assumes that a function is continuous (in the sense of the ε,δ definition of continuity) and that its corresponding curve has tangents to it at every point—or more precisely, that each point on the curve has a *unique* tangent. Of course those assumptions are not true of all functions and their corresponding curves. Many functions are instead discontinuous and many curves contain certain points at which no unique tangent can be described,[20] making them *non*-differentiable (at least at some points). A simple case of a "pathology" that makes a curve non-differentiable is the presence in it of a sharp corner. For example, the graph of the function $y = f(x) = |x|$ (the "absolute value" of x) yields a curve that has such a corner at $x = 0$, where no unique tangent can be assigned to it (rather, any number of distinct tangents can be drawn through the vertex of the corner), and thus the function possesses no derivative for the value of x at that point. Points at which a function is not differentiable are often called "singularities."

On the impulse picture of acceleration, the non-uniform motion generated by percussive impact will be described by a function that is not at all points differentiable. For the instantaneous jumps to different velocities that are introduced by the impulses will "throw corners" into the curve of the motion. When a body traveling at rate r is bumped up to (say) $3/2r$ in an instant, the function "spikes" and becomes, at that one point in the interval of the motion, non-differentiable. So on the impulse picture, non-uniform motion is punctuated with singularities. On the classical contemporary picture, on the other hand, there is an intermediate account to be had: while uniform motion will express itself geometrically as a straight line, *non*-uniform motion will instead trace out a relatively "smooth" curve that has tangents at all points. Thus non-uniformity does not automatically imply the existence of singularities or points at which the function is not differentiable. Only pathological phenomena of the sort that the classical contemporary picture assumes not to exist in nature would have to drive non-uniform motion into non-differentiability.

So when Leibniz denies the uniformity of motion, he is denying that the mathematical structure of motion is that of a straight line. But with his own picture of non-uniform or accelerated motion as an infinity of discrete impulses raining into every interval, the mathematical character of the motion is not a smooth curve either—not even piecewise so. Mathematically, the interval of the motion is *nowhere* differentiable. We shall return to this point later.

4. Charinus's Solution

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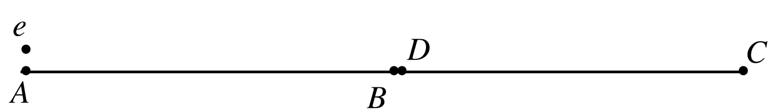
Leibniz's plan to escape from the difficulties of the labyrinth of motion is supposed to exploit an overlooked assumption about motion: namely, that the interval of motion is uniform. The impulse picture he is proposing will require us to discard this assumption and hold instead that the interval of motion of the moving body includes some singularities—indeed, infinitely many such singularities in every subinterval of the motion.

Why will this help? The idea seems to be that the assumption of uniformity is what required us to say that a continuously moving body would traverse a line exactly one point after another and—because the motion is uniform—thereby exhaust the line by moving successively from point to point to point right through to the end. If we can hold motion to be uninterrupted through an interval while denying it to be uniform, however, we may be able to endorse one aspect of the *locus proximus* account (thereby avoiding leaps into remote spaces and expansions into "the whole place" of the interval) and yet avoid saying that an uninterrupted interval of motion must exhaustively decompose space and time into a succession of points and moments. In the following passage we find Charinus as he finishes recapping the perplexity about the composition of continuous motion and then launches the effort at escape that targets the concept of uniformity; his example is again of a "moving point" (this time labeled '*e*') traversing a line *AC*:

> *Ch.*: [...] And so the line will be composed of points, since the moving point traverses the line by going through every single one of these points that are continuously immediately next to each other. But for a line to be composed of points has been demonstrated to be absurd. Since, on the other hand, *uniformity* cannot be denied in place and time considered in themselves, it therefore remains for it to be denied in motion itself. And in particular it must be denied that another point can be assumed immediately next to the point *D* in the same way that the point *D* was assumed immediately next to point *B*.



> *Pa.*: But by what right do you deny this, since there is no prerogative in a continuous uniform line for one point over another?

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> *Ch.*: But our discussion is not about a continuous uniform line, in which two such points *B* and *D* immediately next to each other could not even be assumed, but about the line *AC* which has already been cut into parts by nature; because we suppose change to happen in such a way that at one moment the moving point will exist at the endpoint *B* of one of its parts *AB*, and at another moment at the endpoint *D* of the other part *DC*. [...][21] I deny, therefore, that another point could be assumed in the line *DC* immediately next to *D* for I believe that no point should be admitted in the nature of things unless it is the endpoint of something extended.
>
> (A VI,3,563-4)

Thus Leibniz ties his ontology of points as modes to the idea of non-uniformity: there can be no third point assumed immediately after *B* and *D*, lest the continuum be composed from minima, and so the line cannot have points assigned in it uniformly in that fashion. The discussion of non-uniformity resumes with Pacidius demanding an account of the nature of motion itself that will explain why points are assigned non-uniformly in the line as Charinus proposes, since it is motion that divides the continuum and actually assigns points to the line. Charinus's answer brings to light the central thesis in Leibniz's theory of motion—indeed the very centerpiece of his solution to the paradoxes of the continuum.

> *Pa.*: This nonuniformity which you have established in motion must be explained, since it is from this that the nonuniformity in the division of the line is to be derived. [...][22]
>
> *Ch.*: Then what if we say that the motion of a moving thing is actually divided into an infinity of other motions, each different from the other, and that it does not persist the same and uniform for any stretch of time?
>
> *Pa.*: Absolutely right, and you yourself see that this is the only thing left for us to say. But it is also consistent with reason, for there is no body which is not acted upon by those around it at every single moment.
>
> (*with omissions*; *italics added*; A VI,3,564-5)

Thus with the hypothesis of non-uniformity that was instituted to solve the paradoxes now explained kinematically as the structure belonging to infinitely divided motion (and further corroborated by Leibniz's impulse dynamics), Charinus is given the moment to announce the main conclusion of the new analysis of the structure of motion:

> *Ch.*: So now we have the cause of the division and the nonuniformity, and can explain how it is that the division is arranged and the points assigned in this way rather than that. The whole thing therefore reduces to this: at any moment which is actually assigned we will say that the moving thing is at a new point. And although the moments and points that are assigned are indeed infinite, there are never more than two immediately next to each other in the same line, for indivisibles are nothing but bounds.
>
> (A VI,3,565)

Motion can neither be a purely discrete series of points, as if it were just so many grains of sand, nor can it be purely continuous like an undivided geometrical line. Charinus's proposal tries to find a middle path between pure discreteness and pure continuity: Motion across an interval is divided, not into points, but rather into finer intervals of motion, which are themselves further divided into still-finer subintervals, and so on *ad infinitum*. The motion of a moving thing is thus actually divided into an infinity of other motions, each different from the other, and it does not persist the same and uniform for any stretch of time. The endpoints of those intervals of its infinity of distinct motions assign the actual moments in the continuum of time and the actual points in the continuum of space. At any moment that is actually assigned, the moving thing is at a new point, and the transition of the moving thing from one interval in the line to the next does occur by a single step—by a "leap"—from an assigned endpoint to the *locus proximus*, the indistant but distinct beginning point of the next interval. But the moving thing never omits or leaps over spaces of any size, finite or infinitesimal, since it passes through each and every point in the interval along the way; nor does it expand to fill the whole space of its motion in a moment. And the motion is never dissolved into a mere succession of discrete points, for although the moments and points that are assigned by the passage of the moving thing are indeed infinite, there are never more than two immediately next to each other in the same line. Rather, the divisions or singularities in motion that produce the pairs of immediately neighboring points are always *densely ordered* in the interval, and any *two* pairs will always be separated by a subinterval of the motion. Indivisibles belonging to these pairs remain nothing but bounds, the ends or beginnings of the submotions into which the whole is divided by the actions of impulse forces on the moving body.

5. A First Philosophy Of Motion?

Do we truly have here in Charinus's proposal an answer to the difficulties concerning the structure of motion? I believe we do have an answer (if not an entirely successful one[23]), though I confess that it took me an inordinately long time to see how to understand this proposal. Partly I was slow to understand the account simply because I had failed to observe how carefully Leibniz isolates the con

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cept of uniformity as the point of entry for his new analysis. I hope the passages I have selected here from Leibniz's discussion in the *Pacidius* have made it obvious that this is in fact what he does. But there is something more than my slowness to see the obvious that held up my understanding. I could not see how this proposal could amount to a coherent and positive description of the structure of the interval of motion and not be just a rather *ad hoc* denial of the particular claims about motion, space and time that already proved to lead into absurdities.

Perhaps the "naive" conceptions of motion do indeed lead to paradox as Leibniz suggests. And perhaps no contradiction will arise from Charinus's carefully circumscribed proposal. Still, that proposal seemed only to be a statement of what motion could *not* be: it could not be organized in the ways that provably give rise to the paradoxes of the composition of the continuum.

Such is the counsel of despair for a first philosophy of motion. There is a familiar parallel here in the philosophy of set theory. Upon discovery of the paradoxes of the set theory based on a naive conception of set—for instance Russell's paradox that the existence of a set of exactly those sets that are not members of themselves (which is absurd) can be proved—it becomes necessary that the old theory be abandoned in favor of a new one that will not have the problems associated with the naive conception of set. But a merely technical solution to the paradoxes that comes in the form of a handful of axioms with *ad hoc* restrictions on the size and membership of sets will not solve, or even address, the more basic problem of understanding set theory and the universe it describes. Such an *ad hoc* replacement would only be a mystery account whose "answers" to the paradoxes do nothing to offer foundations for a science of sets—not if that science is ever to be a vehicle for understanding set theory and mathematics. If there is to be a first philosophy for set theory and mathematics, it will have to provide a new conception of set that can both justify its axioms and explain why the old paradoxes will no longer arise within it. Likewise, Leibniz's first philosophy of motion cannot rest simply having advanced a critique of the naive conceptions of motion and having issued a series of stipulations that will suppress the paradoxes. It must articulate a new conception that will yield a positive understanding of the nature of motion—one that will illuminate the elements of Charinus's proposal from the inside and will explain why the paradoxes no longer arise on this new theory.

The lesson of set theory also turns out to be an optimistic one, for there is indeed a positive conception of set that may be up to the task of illuminating a satisfactory theory of sets from the inside (I have in mind here the "iterative conception" and ZF set theory[24]). On a parallel note of optimism, I think there is indeed a positive conception of the interval of motion that can underpin and explain Leibniz's new theory. (Or, more cautiously, there is a concept that can *start* to clarify these Leibnizian matters; even the incomplete support offered to ZF set theory by the iterative conception of set achieves a far higher standard of conceptual clarity than what we can offer here to make sense of Charinus's proposal.) Also, just as in set theory it turns out that the positive conception of set was probably right there in Cantor, the positive conception of the interval of motion—and indeed of the wh

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ole Leibnizian universe, for he will eventually extend his account of motion to include all of space, time, and matter —is right there in Leibniz, too. Or, anyhow, there are sufficiently intriguing hints to make a case for it.

6. Charinus's Interval, The Folded Tunic And The Koch Curve

Charinus's proposal about the interval of motion is elaborated substantially in a later passage in the dialogue in which Pacidius introduces a striking image of the divided continuum in general, of which the interval of motion is an instance; call it the *folds passage*:

> *Pa.*: Accordingly the division of the continuum must not be considered to be like the division of sand into grains, but like that of a sheet of paper or tunic into folds. And so although there occur some folds smaller than others infinite in number, a body is never thereby dissolved into points ôr [*seu*] minima. On the contrary, ... although it is torn into parts, not all the parts of the parts are so torn in their turn; instead at any time they merely take shape, and are transformed; and yet in this way there is no dissolution all the way down into points, even though any point is distinguished from any other by motion. It is just as if we suppose a tunic to be scored with folds multiplied to infinity in such a way that there is no fold so small that it is not subdivided by a new fold ... And the tunic cannot be said to be resolved all the way down into points; instead, although some folds are smaller than others to infinity, bodies are always extended and points never become parts, but always remain mere extrema. (A VI,3,555)

There is a lot to explore here, but we shall focus on only a few points. The image of the tunic "scored with folds multiplied to infinity" provides a heuristic for thinking about the structure of the divided continuum that is perhaps easier to grasp—one whose "shape" is more easily imagined—than Charinus's account of the assignment of points in space and moments in time falling within a linear interval of motion. Just as the interval of motion is divided by differing submotions in such a way that it contains intervals within intervals *ad infinitum*, the tunic is scored with folds in such a way that it contains folds within folds *ad infinitum*. But while this assigns infinitely many points and moments into every interval of motion and infinitely many creases into every part of the tunic, no matter how small, the interval is not thereby dissolved into a powder of points, nor is the cloth of the tunic reduced into a mere collection of creases. Rather, the interval and the tunic are so shaped that they are "nowhere the same and uniform through any stretch," and display structure on all scales of magnification.

Not only does the discussion of the folded tunic provide a heuristic for thinking about the interval of motion in terms of its shape or structure, but also, from a contemporary perspective, it allo

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ws us to identify the interval of motion itself and the divided continuum in general as natural prototypes for a class of geometric objects known as *fractals* that might indeed seem to occupy a space between the purely discrete and the purely continuous. Fractal mathematics was not yet elaborated in Leibniz's time, but the structure of the interval of motion, as Leibniz describes it, appears to display the very properties that fractal mathematics was later developed to study. Just as the ancient concept of continuity already under discussion in classical texts came to be explored two millennia later in the field of topology,[25] the conception of the interval that is being newly expressed in the *Pacidius* belongs naturally to a mathematics lying centuries away in its future. And it is the conception of a structure that will eventually be called fractal which underlies Leibniz's first philosophy of motion.

The field of fractal geometry has taken shape prominently in recent decades due to the pioneering work of Benoit Mandelbrot.[26] As a late-twentieth-century discipline, the study of fractals has largely been developed within the framework of set theory and the point-set approach to the mathematical continuum in particular. This requires a cautionary note. Much of fractal mathematics will involve definitions and concepts that were in no particularly illuminating sense available to Leibniz. Moreover, Leibniz's own fundamental views of motion and the continuum are at odds with the point-set approach to those subjects. His repeated denials that "infinite aggregates" could constitute "true unities" (cf. A VI,3,503, A VI,6,157) immediately excludes from consideration those concepts that make sense only in the setting of contemporary set theory with its axiomatic demand that there be sets with infinitely many elements. Also, his insistence that points are only modes of the continuum and not independent elements of it would appear to be contravened by the basic assumptions of the point-set analysis of the continuum. Like so many concepts in contemporary mathematics, however, the concepts of fractal mathematics that we shall use to illuminate Leibniz's theory of motion do not essentially presuppose the point-set foundations from which they are typically developed, and they can be cashed out independently of it in terms that respect Leibniz's fundamental views. In what follows our account will be conducted in terms that accord with Leibniz's philosophy, and the point-set analysis will not be presupposed.

The two concepts that we shall borrow from fractal mathematics—*fractal dimension* and *complexity on all scales*—are its two most central notions, and they will be worth some recounting. Then in considering how they apply to a simple fractal curve discovered in 1904 by Helge von Koch, we shall provide the connection between fractal mathematics and Leibniz's theory of motion. For the Koch curve itself offers a clear match for the structure of the interval of motion as Charinus describes it and as it is elaborated in the image of the folded tunic.

A fractal structure is defined in terms of two distinct concepts of dimension. The first is the familiar topological concept according to which structures are zero-, one-, two-, three- or in general *n*-dimensional, where this roughly amounts to the following fact: an *n*-dimensional continuum can be disconnected or separated into disjoint parts by the removal of some number of *n-1* dimensional p

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ieces of it. To cut a one-dimensional line into two parts it suffices to remove a zero-dimensional point, or in the case of a circle the removal of two points will disconnect it into a pair of disjoint arcs. Likewise, the two-dimensional square can be cut in two by one-dimensional lines, and the three-dimensional cube by two-dimensional planes, and so on.

The second concept of dimension is fractal dimension—now often simply identified with Hausdorff dimension[27]—which, as it happens, can assign a fraction as the order of a dimension rather than an integer. A structure is a fractal when its fractal dimension exceeds its topological dimension. We can think of fractal dimension as a measure of how effectively a structure fills a Euclidean space of the more familiar kind: the higher its fractal dimension, the more effectively a structure fills in space. To take one vivid example of a classical fractal structure with non-integer dimension, it is possible to define a "dust" of infinitely many isolated points lying in a linear interval that has fractal dimension 0.630923; the dust fills in space in the interval better than the zero-dimensional point but less effectively than the one-dimensional line. The topological account would assign dimension zero to such a dust, not distinguishing it dimensionally from a point. Classic instances of fractal structures are objects that fall dimensionally "in between" points and lines, or lines and surfaces, or surfaces and solids.

Full-dress Hausdorff dimension itself involves rather heavy technical apparatus, and in practice for most cases one typically substitutes simpler (less general) definitions to compute values for fractal dimension. Moreover, the apparatus for assigning Hausdorff dimension is deeply invested with set-theoretic constructions of the sort that would be anathema to Leibniz.[28] Yet both the computational and conceptual difficulties with relying on Hausdorff dimension to explore Leibniz's account can be laid aside by turning to a simple version of fractal dimension known as "scaling dimension" that can be defined entirely in algebraic terms and which still captures the basic insights of fractal geometry. That is the approach we shall follow here.[29]

The concept of fractal dimension and the general sense in which it yields a measure of how effectively a structure fills space can be illuminated by paying attention to the properties of a special subclass of fractal structures, the so-called self-similar ones, and their "scaling" properties. (As we shall see later, the particular concept of fractal dimension that we are going to explore in this way is also quite close to what is suggested by Leibniz's own remarks.) A structure is (strictly) self-similar if it can be broken down into arbitrarily small pieces each of which is a scaled-down replica of the entire structure.[30] Likewise, with sufficiently many copies of a self-similar structure one can assemble a larger-scale replica of it. It turns out that there is a crucially important relation that holds between (i) the number of equal copies required to assemble a replica and (ii) the scale of that assembled replica relative to the individual copies. For example, two copies of a one-dimensional line segment suffice to assemble a line segment twice the size of the original.[31] It requires *four* copies of a two-dimensional square to assemble a square twice the size of the original; and *eight* copies of a three-

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dimensional cube are needed to assemble a cube twice the size of the original. In general for a d-dimensional hypercube one needs 2^d copies in order to assemble a further one twice the size. Now generalizing to cover any scale increase in size, we find that it requires $c = a^d$ copies to make a replica a times the scale size of the original copies. (The factor a is the *assembly ratio* that tells us how large the assembled replica is relative to each equal copy.) In the equation

(1) $$c = a^d,$$

the term d is understood to express a measure for the dimension since it agrees with topological dimension in familiar example of geometrical shapes and provides a well-defined value in more exotic cases as well. Taking seriously the idea that d is a measure of dimension, we turn around equation (1) and introduce a definition of scaling dimension by solving for d. To obtain d from $c = a^d$ we take logarithms on both sides:

(2) $$d = \log c / \log a.$$

Thus the scaling dimension of a structure that requires c copies of itself to make one a times the size of the original is $d = \log c / \log a$. And perhaps now it is not so surprising that the dimension d will turn out to be a fraction for some structures.

To take a classic example, Cantor's "ternary set" is a structure whose size can be tripled ($a = 3$) by assembling two copies of it ($c = 2$). Thus the scaling dimension of the Cantor set is $d = \log 2 / \log 3$, or $d \approx 0.630923$. The Cantor set here is in fact the very example of a "dust" mentioned before whose fractal dimension falls between zero and one. Structures whose fractal dimension is precisely the same as their topological dimension make up what turn out to be the vanishingly small species of familiar "smooth" structures like Euclidean planes or spheres. On the other hand, structures whose fractal dimension exceeds their topological dimension are paradigm fractals—objects that are from our point of view very "rough" or "irregular" in shape but from the fractal point of view entirely typical.

That last point takes us to the second prominent characteristic of fractals, and one intimately connected with scaling dimension and the "roughness" of fractal structures. It is the fact that fractals typically exhibit *structural complexity on all scales*. To see the significance of this, first consider the typical case of an ordinary non-fractal curve. Although perhaps describing a complex figure when viewed from a wide scale, if we nail down some small interval of the curve, typically the degree of complexity found in the interval is less than that of the whole. As we continue to "zoom in" on successively smaller intervals what we tend to find are increasingly smooth and graceful pieces of the curve which, taken individually, look more and more like straight lines. Pushing this process into th

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e limit, the small intervals come to look exactly like straight lines that describe tangents to the curve itself. This is of course just an image of *differentiation*: indeed, in Leibniz's calculus every small interval "in the limit" is itself an infinitely small straight line connecting the endpoints of an infinitely small interval, and the whole curve itself is described as the boundary of a polygon with infinitely many infinitesimal sides each of which, when extended in both directions, yields a tangent to the curve. Ordinary non-fractal curves are thus "rectifiable" or differentiable. Or they are at least *piecewise* differentiable, even if they should include (say) some sharp corners that never settle out into straight lines no matter how small we take the intervals containing them.

By contrast, fractal curves typically are *not* rectifiable or differentiable, not even piecewise so. For these "monster curves," no matter how small a scale you pick, the pieces enclosed in the small intervals continue to display kinks and changes of direction and *never* settle into straight lines. Under such conditions it is possible that there be *no* places on the curve at which tangents can be drawn; so instead of the curve eventually becoming "all lines" at some scale of magnification, it may instead simply be (*e.g.*) "all corners" and thus be *nowhere differentiable*. This is one sense in which fractals can display structural complexity on all scales. Self-similar fractals will exhibit precisely the same type and degree of complexity at every scale. Yet even fractals that are not strictly self-similar, but rather incorporate different structural features at different scales, are nonetheless complex "all the way down." (Hausdorff dimension in a sense generalizes scaling dimension, for it assigns the same value for dimension to self-similar structures as scaling dimension does, but it can also assign values for dimension to structures that are not strictly self-similar but have scaling complexity of other types.)

The history of fractal mathematics is often traced well back into the nineteenth century, though by all accounts it begins fully to crystallize at the start of the twentieth century. The subject takes hold decisively in 1904 when Helge von Koch proposes a fractal curve that is to become a classic in the field and which carries the distinctive scaling structure of "folds within folds" that is suggested in the image of the tunic in the *Pacidius*.[32] Koch's curve is constructed recursively from an "initiator" line segment of unit length according to this basic construction step: cut the line segment into three equal pieces, then replace the middle third by an equilateral triangle and take away its base.

[[*typesetter: insert Figure A here*]]

The resulting figure (called the *generator*) is at the next stage subject to the same construction step: take *each* of the resulting line segments, cut it into three equal pieces, replace the middle third by an equilateral triangle and take away its base. Repeat this construction *ad infinitum*; the sequence of polygonal curves constructed in this way converges to a limit figure that is itself the Koch curve.

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[[*typesetter: insert Figure B here*]]

The Koch curve provides a simple example with which to illuminate Charinus's proposal about the structure of the interval of motion, for it provides a very clear bridge between Leibniz's heuristic image of the folds, on the one side, and the concepts of fractal dimension and complexity on all scales, on the other. Starting with its scaling dimension: *four* equal copies ($c = 4$) of the Koch curve can be assembled to form another Koch curve *three* times the size ($a = 3$) of the originals. Our definition of scaling dimension ($d = \log c/\log a$) then yields the following result for the Koch curve:

$$d = \log 4/\log 3 \approx 1.2618$$

Since the Koch curve can be disconnected into disjoint pieces by the removal of a single point, its topological dimension = 1. Thus the Koch curve is a paradigm fractal by the definition of a fractal structure: its fractal dimension strictly exceeds its topological dimension. Notice as well that it is "non-rectifiable": any interval you pick, however small, always contains a sharp corner—indeed, infinitely many sharp corners. The Koch curve is, in a sense now quite clear, a structure that is "all corners."

The Koch curve develops Leibniz's image of the folded tunic to a very high degree of perfection. Every fold in the tunic corresponds to a "major" piece of the Koch curve: each such piece appears relatively smooth or flat at one scale of magnification, but at a finer scale it is revealed to be subdivided by a smaller "triangle," each major sub-piece of which is itself subdivided by a smaller "triangle" and so on *ad infinitum*. (There are fleas on the fleas...) And just as Charinus demands non-uniformity throughout the interval of motion, so no extended piece of the Koch curve will "persist the same and uniform through any stretch." Further, the points assigned at the vertices of the curve—the peaks and feet of the triangles—are always separated from any other such point by an intervening vertex of some triangle. If, along with Leibniz, we understand each vertex in fact to be an aggregate-pair of indistant points lying at the locus of contact between two pieces of the curve, then what we find is that "there are never more than two immediately next to each other in the same line." The "corners" or singularities are densely ordered in this fractal curve. Recent authors in mathematics writing about the Koch curve describe it as a complex of "folds within folds within folds."[33] Had

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Charinus drawn a figure of his non-uniform uninterrupted interval of motion, he could hardly have done better.

As a number of writers on Leibniz have recognized, fractals also make nice embodiments of Leibniz's well-known claims that there are worlds within worlds *ad infinitum* and that every mote however small contains a world of the same infinite variety as the world we find on our own scale of experience.[34] There is an abundance of passages in Leibniz that document this view of the scaling structure of the world; as a representative sample we might consider two. First, from a 1698 letter to Johann Bernoulli:

> I am not joking, but clearly admit, that there are animals in the world as much larger than ours are, as ours are larger than those tiny animals of the microscopists, for nature knows no boundary. And, on the other hand, there could be, indeed, there have to be, worlds not inferior in beauty and variety to ours in the smallest motes of dust, indeed, in tiny atoms. (GM III,553)

And second, from the 1676 *Pacidius* itself, where Theophilus says to Pacidius:

> Well, you have certainly astounded me. Those who claimed that there are infinite spheres of stars in this mundane space, and that there is a world in every sphere, seem to have said something of importance; whereas you show that in any grain of sand whatever there is not just a world, but an infinity of worlds. I doubt if anything could be said that is more splendid than this. (A VI,3, 566)

This doctrine of world within worlds has astounded many audiences through the years, to better and worse effect.[35] Mathematicians today do not read Leibniz, but those of the nineteenth and early-twentieth centuries did, and already in their writings one can find an interest in those nowhere differentiable "monster curves" satisfying the broad outlines of Leibniz's astounding metaphysical doctrine but defying treatment by his calculus.

Recall now Leibniz's new impulse picture of accelerated motion according to which the interval of motion includes infinitely many singularities due to the onslaught of impulse forces that act on the moving body at every single moment. We can see how, and in a sense why, motion displays fractal structure: the rain of impulses ensures that it is "all singularities" and nowhere differentiable. Thus the view of motion as a fractal fits just as it should with the account of accelerated motion that we explored earlier. It is exactly the mathematics needed to flesh out the dynamics of acceleration suggested by Leibniz.

How close, then, is Leibniz to articulating the link between his metaphysics of motion and the fractal mathematics that could serve as its foundations? It does not appear that Leibniz ever raises t

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he higher-order question of what kind of mathematics might be required for his new conception of motion and the divided continuum in nature or for his worlds-within-worlds doctrine; and there is no immediately compelling reason to suppose that he would have identified the correct answer had he considered the matter in such direct terms. Still, fractal mathematics is not entirely beyond Leibniz's mathematical purview, and in fact extraordinary technical insights into fractal mathematics do appear repeatedly in his writings on mathematics and geometry.[36] Benoit Mandelbrot himself describes sampling Leibniz's mathematical writings as "a sobering experience," for "the number and variety of premonitory thrusts is overwhelming"(1983, p. 419). Among Leibniz's insights to impress Mandelbrot, perhaps the most notable were the ideas that he outlines in 1695 in letters to Guillaume François de l'Hospital and Johann Bernoulli concerning "differentials whose exponent is a fraction or an irrational number"(GM III.i,228) and the general technique of fractional integro-differentiation which incorporates the idea of fractional dimension and is now a textbook example of fractal mathematics.[37]

Perhaps even more striking are the insights recorded in a document on geometry titled *In Euclidis* □□□□□ (1679-89?).[38] There Leibniz proposes to tighten some of Euclid's elementary definitions, yet along the way he challenges the classical view of dimension:

> But imperfect is this doctrine, mentioned before, that there are only three dimensions, or in other words that whatever lacks depth and has width, ôr is something intermediate between a line and a solid, is of one and the same dimension. Consequently it must be shown whether the section of a solid is a surface, that is, a magnitude whose section is a line [...] Furthermore, from this it is clear that *surface* is not sufficiently defined, for indeed, as I said, it is not settled that everything which has width and lacks depth is of the same dimension. (GM VI,187)

Surface appears to be "insufficiently defined" on the standard Euclidean account, for it is not trivially true that whatever falls between a (three-dimensional) solid and a (one-dimensional) line must be something whose section is a line. And in fact Leibniz envisages the possibility of an "ascending dimension" intermediate between that of a solid and a simple Euclidean surface and whose section is not a line but a further surface:

> It might be considered whether it is possible that an ascending [*ascendendo*] dimension exists intermediately between a surface and a solid, whose section is a surface, which would not now be the highest dimension but rather it would be possible in turn to assign another section. (GM V, 188)

Leibniz will call the structure belonging to this intermediate ascending dimension an “ascending surface.” Like a solid, the ascending surface will have as its own section a surface, yet it does not thereby automatically belong to the “highest dimension” (i.e. that of the solid); rather, there could be still other sections of a solid that would rank higher in dimension than this ascending surface. Thus it appears that there could be a whole spectrum of ascending surfaces intermediate between a merely two-dimensional surface and a solid. Leibniz naturally then raises the possibility of “descending” surfaces as well:

> Likewise, if an ascending surface could be defined which is the section of a solid, then surely the descending [*descendendo*] can be considered—whether there might not exist such an intermediate between a surface and a line. (GM V,188)

Leibniz’s ascending and descending surfaces and his idea of intermediate dimensions very perfectly anticipate the basic elements of contemporary fractal theory, and they fit neatly with the examples already in play in our discussion. The Koch curve has a dimension intermediate between a line and a surface; Leibniz’s folded tunic (disregarding, as Leibniz does, the depth of the cloth) would fall intermediately between a two-dimensional surface and a solid. These new ideas about geometry will not be thoroughly elaborated for another two hundred years or more, but from vantage point afforded by *In Euclidis* □□□□□ it appears that the doors to fractal geometry are already being opened.

Leibniz may well not see the connection between his new account of motion and his thoughts about the geometry of intermediate dimensions and thus not weave them together into a single comprehensive theory. But by any measure it should be clear that in his account of the non-uniform uninterrupted interval of motion in *Pacidius Philalethi*, Leibniz does indeed have a new positive conception to illuminate the details of his theory from the inside, and it is a conception that (in later hands) will prove to be enormously mathematically and scientifically fecund. For in the *Pacidius* motion is fractal in structure, and this fractal conception is the centerpiece in Leibniz’s first philosophy of motion.

7. The Nature Of Motion

Up to now I have been holding aside a few critical points concerning Leibniz’s theory of motion in the *Pacidius*, and in this concluding section I mean to consider them in combination for the light they cast together upon his first philosophy of motion. Leibniz’s analysis of motion as fractal in structure faces a serious difficulty: the account of minima as modes of extended intervals appears to be incompatible with his analysis of the infinite division of motion. Two other strands of thought in Leibniz’s writings of this period—one in the *Pacidius* concerning the fundamental nature of motion, the

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other in the associated 1676 piece “On Infinite Numbers” concerning the role of imagination in perceptual experience—reveal more of the architecture of his account and suggest a radical view of Leibniz’s early metaphysics of motion that could offer a solution to the difficulty. Yet I shall reject this radical view and the solution it offers. As I see it, Leibniz’s account of motion *is* problematic. Understanding the problem and the reasons for rejecting the radical view of Leibniz’s metaphysics of motion that could solve it, however, will provide a fuller view of his actual thought about motion as well as a clear vantage point from which to consider the contents of his early metaphysics of the corporeal world more generally.

First, to state the difficulty. On the fractalist theory of motion, each singularity in the interval of motion is a place at which the motion is actually subdivided into smaller submotions, and according to Leibniz each such locus of division is in fact an aggregate pair of indistant points. The moving body makes a “leap” from the end of one subinterval to the beginning of the next, though not in the objectionable sense of a leap that omits some intervening space (cf. A VI,3,567); and every leap marks a change in the motion of the moving body that occurs at the boundary between distinct submotions. As has been noted already, Leibniz’s theory constitutes an infinitary version of the *locus proximus* account of motion: any motion across an interval into a remote space contains a densely ordered infinite series of singularities, and so also a densely ordered series of unextended leaps between the indistant ends and beginnings of its various submotions. The ends and beginnings of those submotions are not independent elements from which the motion is composed, however, since they are always minima and Leibniz’s ontology of minima requires that they be only modes of extended things, the extrema or endpoints of intervals. The ends and beginnings of the submotions do not “pre-exist” but rather depend for their own existence on the extended intervals that they bound; the intervals are ontologically prior to the endpoints. Yet it becomes difficult to see how there could be any extended intervals of motion on the *Pacidius* account of motion. If motion has the fractal structure of the folded tunic, it would appear in the end to be “all corners” or “all singularities”—that is, “all leaps” between indistant *loci proximi* with no ontologically prior extended intervals left to possess the beginnings and ends of motion as their endpoints.[39] And this appears to be a deep problem in his metaphysics of motion.

What we find here, I believe, is a tension in Leibniz’s thought between two competing conceptions of motion. On one side, an interval of motion is conceived as consisting of an infinite series of discrete leaps between indistant *loci proximi*. On the other side, an interval of motion is conceived as always divided into finer and finer subintervals and never resolved all the way down into minima or “points” of motion. Leibniz’s fractalist account strives to reconcile those conceptions by assigning the leaps to occur always between the indistant endpoints of neighboring subintervals (that is, at the places at which motion is divided into submotions), and by understanding the divisions of an interval of motion to be distributed across an infinite descending hierarchy of distinct scales so that a

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t no point in the assignment of leaps—at no single scale in the scaling structure of divisions—is motion ever found to be resolved into a mere "powder of points." The continuity of the extended interval is thus gradually pushed towards convergence with the discreteness of the underlying series of unextended, punctual leaps, yet somehow without actually reaching a limit state of a division into points. Considered "from the inside" or from the point of view of any one scale within the fractal hierarchy, no problem seems to arise, for the scaling structure does not include any final "limit scale" of ultimate resolution within its ranks. But considered "from the outside" where *all* the scales and *all* the divisions in the structure are given at once, a resolution of motion into a powder of points (or leaps) seems to be inevitable—for having cut back every finite interval into finer and finer parts without end, what extended intervals could now remain?

It could not be clearer in the major passages in which the fractal account emerges that Leibniz means to stay fast in his loyalties to those two different conceptions of motion even as they seem to come into conflict. The folds passage concludes:

> And the tunic cannot be said to be resolved all the way down into points; instead, although some folds are smaller than others to infinity, bodies are always extended and points never become parts, but always remain mere extrema. (A VI,3,555)

And another important passage, in which Leibniz projects the fractal theory of motion across the whole fabric of nature, likewise concludes with the author's gaze turned back to the source of concern:

> It will be worthwhile to consider the harmony of matter, time and motion. Accordingly I am of the following opinion: there is no portion of matter which is not actually divided into further parts, so that there is no body so small that there is not a world of infinitary creatures in it. Similarly there is no part of time in which some change or motion does not happen to any part of a body or point whatsoever. And so no motion stays the same through any space or time however small; thus both space and time will be actually subdivided to infinity, just as a body is. Nor is there any moment of time that is not actually assigned, or at which change does not occur, that is, which is not the end of an old or beginning of a new state in some body. *This does not mean, however, either that a body or space is divided into points, or time into moments, because indivisibles are not parts, but the endpoints of parts; which is why, even though everything is subdivided, it is still not resolved all the way down into minima.* (A VI,3,565-6; italics added)

Leibniz is doing his best here to have it both ways in his attempt to reconcile the discrete and the continuous, and the designs he throws up in his efforts to achieve this end contain philosophical and mathematical insights of the highest order. But the not all the conflicting demands on his theory of

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motion can be satisfied at once, for the ontology of motion that would compose it from actually infinitely divided submotions is inconsistent with Leibniz's metaphysics of minima as the ontologically dependent extrema or endpoints of extended intervals. The new fractal conception in fact cannot prevent the consequence that motion (or body, space or time) will be "resolved all the way down into minima" on Leibniz's account—his own assertions to the contrary notwithstanding.[40] Eventually some element of this account must yield to the others if a coherent metaphysics of motion is to be secured.

This conflict in Leibniz's thought may not finally be resolvable. But I think it proves to be highly illuminating to consider how his analysis of motion is further articulated, for it suggests a path along which his theory could progress towards coherence and brings the content of his metaphysics of motion into sharp relief. Also, exploring the potential refinements to Leibniz's account helps to frame an important question concerning the general interpretation of his metaphysics during this period.

It is possible for Leibniz to make some headway towards resolving the difficulties that face his theory of motion by dividing the properties of motion into those that belong to its manifest image (so to speak), and those that belong to it as it is in itself or behind all appearances. The conception of motion as a "divided continuum" whose internal scaling structure is unfolded in scales might be understood to apply to the *phenomenon* of motion across an interval of space—motion as it appears to us—while the conception of motion as the resultant of an infinity of discrete leaps between indistant *loci proximi* is reserved for the reality that ultimately underlies that phenomenon. The full analysis of motion would thus develop two sides to its account: motion as it appears in perceptual experience and motion as it is in itself. In perceptual experience, motion appears to consist in extended intervals whose fine structure can always be revealed to contain further subintervals, *ad infinitum.* In itself, motion consists in a densely ordered infinite series of discrete, unextended leaps. The leaps come to be manifest in experience as the "singularities" in motion at which it is accelerated and thereby perceived as being divided into distinct submotions, but not all the leaps or divisions are consciously perceived.

Although this two-sided account of motion is not on display in the *Pacidius* itself, Leibniz offers a striking picture of the activity of the mind in constructing a limited experience of reality in a piece written in April of 1676, just months prior to the *Pacidius*, and titled "On Infinite Numbers." There he comments on the role of the imagination in concealing from sense perception infinitely many "irregularities" and "inequalities" that exist in things (cf. A VI,3,499), suggesting that the world being perceived is too complex, too finely grained and too rapidly changing for all its details to be captured in memory and sustained in conscious perception. Most of those irregularities, though momentarily "sensed by our consciousness," are immediately "forgotten," and the content of sense perception is filled out in our memory by a "thought of uniformity" provided by the imagination, r

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esulting in a sort of cognitive illusion of uniformity—“because of this memory,” Leibniz writes, “we ascribe the name uniformity”(*ibid.*). Sense perception thus sustains within consciousness an experience of a world of finite complexity and piecewise continuity, though a complexity that can be understood to increase without bound at increasing scales of resolution.[41]

Those ideas from “On Infinite Numbers” can be naturally interwoven with the *Pacidius* analysis of motion to yield a more comprehensive account that allows for motion to be a discrete quantity while appearing to us to be continuous. Motion across an interval consists in a densely ordered series of leaps that are presented in experience as a continuum divided into subintervals by discrete changes in motion. The mind negotiates between reality and experience by plastering over most of the changes in motion—most of the leaps—with an appearance of uniformity, or the appearance of stretches through which the motion seems to “persist the same and uniform,” and only a limited portion of the actual underlying causal structure is ever manifest in conscious experience. Still, no particular change in motion is so subtle that it cannot in principle be perceived; rather, all the changes are momentarily sensed, though only a finite number are sustained in memory. As the observation of any actual motion is refined in perceptual experience, say by the aid of instruments, the manifest image of uniformity gives way to a yet more variegated image of a motion divided into many submotions. But at no single scale of resolution in the unfolding of reality within experience is the phenomenon of motion displaced altogether by a series of unextended leaps; the actual infinity of singularities in any given interval is always given in perceptual experience within the limited framework of a *potentially* infinite scaling structure of intervals within intervals—*constructivism* about the infinite operates within the domain of consciousness—in which the leaps are distributed across scales and always appear as boundaries between extended subintervals of motion. In themselves, however, the leaps are neither intervals nor endpoints of motion: they are rather punctual actions of substances (a point that will be clarified shortly below). This, I think, is the most promising way to understand Leibniz’s account as it attempts to reconcile two competing conceptions of motion. Motion is metaphysically founded in an infinity of discrete leaps between *loci proximi* and cognized constructively as a divided continuum with limitless scaling complexity in which a complete resolution into minima is never reached.

If in the *Pacidius* Leibniz does not articulate his account of motion in just that way, he nonetheless makes it clear that the analysis of the nature of motion is to be a reductive one that gives priority to the idea of motion by a leap and locates the reality of motion in its causes. Near the close of the dialogue Leibniz rejects the idea that motion is an explanatorily basic property of individual bodies. The true causal structure to be found in reality consists in “action,” and Leibniz argues that his analysis of change and motion shows that bodies in motion do not act:

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Pa.: But I would like you to notice something else, that this demonstrates that bodies do not act while they are in motion.

Th.: Why is that?

Pa.: Because there is no moment of change common to each of two states, and thus no state of change either, but only an aggregate of two states, old and new; and so there is no state of action in a body, that is to say, no moment can be assigned at which it acts. For by moving the body would act, and by acting it would be changed ôr acted upon; but there is no moment of being acted upon, that is, of change or motion, in the body. Thus action in a body cannot be conceived except through a kind of aversion. If you really cut to the quick and inspect every single moment, there is no action. Hence it follows that proper and momentaneous actions belong to those things which by acting do not change.

(A VI,3,566)

Since the idea of a momentaneous state of change has been refuted, and any action by a body requires change, it follows that there can be no action by a body in any moment. Thus if there is to be any "proper and momentaneous" action in the world at all it must belong, Leibniz suggests, to "those things which by acting do not change"—the slip into the plural perhaps foreshadowing a later theory according to which there is a plurality of beings whose activities ground the changes that occur in particular bodies.[42] The official theory of the *Pacidius*, however, will acknowledge only the *summa rerum* as a source of action without change.

Leibniz does not theorize about the nature of action in the *Pacidius*,[43] nor does he detail the role of action in his physics, although he will go on to explore both at great length in later writings.[44] Yet it is reasonably clear what role action plays in the *Pacidius* account of motion. The "proper and momentaneous" actions at the root of motion are the leaps between indistant points (or perhaps one should only say that those actions constitute what is ultimately "real" in those leaps, since the exact nature of the reduction of motion to action is left unexplained). This has some consequences for the metaphysics of motion. Since the leaps are effected by divine intervention—the body is being transcreated by God—it seems that the changes in motion, i.e. the actions of accelerative forces, are not the operations of moving bodies but rather those of the *summa rerum*. And this immediately suggests that motion is not a real property in bodies at all, but is merely a positional phenomenon that results from God's creative activity. In discussing an example of a body *e* moving between two immediately neighboring spheres *B* and *D*, Leibniz writes:

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> I do not think that we can explain this better than by saying that the body *e* is somehow extinguished and annihilated at *B*, and is actually created anew and resuscitated at *D*, which you may call by the new but very beautiful name *transcreation*. Moreover, although this is indeed a sort of leap from one sphere *B* into the other *D*, it is not the kind of leap we refuted above, since these two spheres are not distant. (A VI,3,567)

The doctrine of transcreation casts the reality underlying the phenomenon of motion in a new light. Since the analysis of change rules out "middle moments" or "momentary states of change," it seems that there will be no explanation for why a given change occurs—for why something changes from one state to another. Or, more exactly, the cause or explanation for the change is not to be found simply in the states that are alternated in the change, but in the action of some abiding substance that underlies the change and both annihilates the earlier state and creates the later one. Leibniz presses the point:

> And this is the thing, finally, for whose sake I have indulged in so many artifices of reason, namely to force you at last into acknowledging so momentous a truth. I add only that transcreation is not what disturbs you: for to say that a thing ceases to exist here, but begins to exist there, with the transition or middle state eliminated, is the same thing as saying that it is there annihilated, there regenerated. And if one person were simply to say that the thing ceases to be in its earlier state and now begins to be in another one, someone else might say that it was annihilated in the earlier state and regenerated in the later one. Whichever of the two you accept, no distinction can be observed in the thing itself, but only in the fact that the former way of putting it conceals the cause, and the latter brings it out. But no cause can be conceived for why a thing that has ceased to exist in one state should begin to exist in another (with the transition eliminated, of course), except a kind of permanent substance that has both destroyed the first state and produced the new one, since the succeeding state does not necessarily follow from the preceding one. (A VI,3,567)

With no necessity connecting the earlier and later states, there is nothing in bodies to produce change or even to secure their own existence through it. A moving body therefore cannot maintain its motion from one moment to the next of its own accord. Indeed, Leibniz has the interlocutors explicitly consider, on their new theory of motion, the status of the ancient axiom "Whatever is once set in motion will always move the same way unless it meets with an obstruction," and Pacidius concludes:

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> This axiom is completely overturned by the doctrine of motion we have developed up to this point. For motion stops altogether, and does not last for any time however small, but at any moment you please the lifeless is resuscitated by the aid of a superior cause. (A VI,3,568)

The point here is quite general. The possibility of change requires some permanent substance or “superior cause” that transcreates the changing thing from one moment to the next and determines what new states it will possess. Apart from the action of some superior cause, there could be no change at all. Since motion is a kind of change, motion finally consists in a series of actions that are to be ascribed not to individual bodies but to the one permanent substance which by acting does not change, namely, God.

Thus the *Pacidius* articulates two layers of philosophy concerning motion. Motion is fractal in its composition; and it is the result of the creative activity of God in its fundamental metaphysical nature. Still, for all the fine detail that has been uncovered in Leibniz’s account, a crucial point about the metaphysics of motion remains unclear in the *Pacidius*. Is there, finally, such a thing as motion? To resolve the tension between the description of motion as a divided continuum and the description of it as an infinite series of discrete leaps, it has been suggested that the appearance of continuity in motion involves something imaginary and belongs to the mind, whereas the theory of unextended leaps is to be reduced to an ontology of substantial activity and so the leaps themselves are to be identified as belonging to fundamental reality. One might wonder now whether this explication of the phenomenon of motion and the reductive analysis of its causes have not simply eliminated motion from the Leibnizian universe. Is there actually nothing in the world but the perceiving mind and the actions of an unchanging divinity? Perhaps there is a darker meaning to be taken from Leibniz’s statement that on the doctrine he has developed “motion stops altogether,” and his first philosophy of motion has ultimately become a form of nihilism. Moreover, the theory of motion itself is deeply interwoven with Leibniz’s theory of matter; after all, for Leibniz at this time bodies are individuated by their motions,[45] and indeed the account of motion in the *Pacidius* is intended to generalize to become the account of the whole of corporeal reality (that is the point of Leibniz’s remarks at A VI,3,565f. about the “harmony of matter, time and motion”). Nihilism about motion would therefore mean the end of the corporeal world in Leibniz’s metaphysics.

A radical view of the metaphysics of the *Pacidius* would simply be to embrace the nihilist reading of motion and corporeal reality, and to replace a “realistic” account of the external world in Leibniz’s metaphysics with an “idealistic” one. If that is the upshot of Leibniz’s first philosophy of motion, however, he has said nothing in the *Pacidius* to prepare his audience for such a revelation, nor indeed does he show any evident signs of being conscious of it himself. Even in his writings of the surrounding months, the existence of matter and motion never appears to be put into question. The possibility of a wholesale elimination of corporeal reality in favor of a theory of incorporeal subs

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tances seems not to be on display in his texts and indeed not to be on his mind during this period. I doubt that it is intended by Leibniz to be a consequence of his philosophy in the *Pacidius*. A more natural reading of Leibniz's own view, I would suggest, is that the actions of the superior cause somehow give rise to a world of bodies moving in a plenum of matter—to be sure, a world that depends constantly for its existence on divine activity, but a world whose existence is to be understood "realistically" rather than "idealistically" or as a construction from thoughts in the minds of created substances. Corporeal reality is what *results* from God's creative activity, and motion is what results from God's infinitely many discrete, punctual acts of transcreation. This corporeal world is infinitely complex and fractal in its structure, but in the sense perception of finite minds it always appears as only finitely complex and piecewise continuous, though in fact there is no end to the non-uniformities and discontinuities in nature that could be eventually made manifest in conscious experience.

Of course the texts will not speak conclusively in favor of this understanding of the final ontological status of motion and matter in the *Pacidius*; Leibniz simply does not pose the question of eliminativism sharply enough for his writings to decide the issue so certainly for his readers. Also, the "realistic" understanding of motion that I propose will not yield within Leibniz's philosophy an unproblematic metaphysics of motion and the corporeal world in general. The basic difficulty for his account still remains in force. Motion and matter are supposed to be everywhere fractally divided quantities—folds within folds *ad infinitum*—without thereby being resolved into minima, and yet that seems not to be possible. With the folds completely assigned, as they must actually be in metaphysical reality itself, the fractal quantity is "all corners"—all singularities, all leaps—and there can be no ontologically prior intervals to which the minima can be assigned as mere endpoints. Leibniz's account of the constructive role of the imagination in cognizing the infinitely complex reality that underlies the phenomenon of motion can preclude a resolution of motion into minima only as it appears within conscious experience, and this leaves the real difficulty untouched. The constructivism or potentialism about the infinite that is implicit in Leibniz's account of our mental operations may well be in order in explaining the content of our experience of reality, but it has no place in the account of reality as it is in itself. For, like reality itself, the fractal division of motion and matter is non-constructive, an *actual* infinite, in which all the divisions are actually assigned and the resolution into singularities or leaps is complete, independently of the capacity of the mind to represent this in consciousness.

The resilience of that difficulty on a realistic understanding of Leibniz's account of motion and the corporeal world might seem to recommend an idealist construal of his metaphysics. If, as the idealistic construal would have it, the account of motion were to be divided cleanly into facts about the constructive cognitive activity of the imagination in finite minds and facts about the actually infinite series of discrete punctual actions by the *summa rerum*, with nothing further left to stand as a real quantity that actually possesses a fractal structure of folds within folds as it is in itself, then the d

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ifficulty for his theory would be resolved. Idealism thus provides Leibniz with an answer of sorts and a coherent way of completing his first philosophy of motion. Looking ahead to his much later writings in which there is considerable textual evidence for ascribing an idealist account of the corporeal world to Leibniz's philosophy, one might be inclined to see idealism as the secret core of the *Pacidius* philosophy as well.[46]

To accept such an idealist view of the *Pacidius*, however, would be to sacrifice the purpose and content of Leibniz's first philosophy of motion in the name of saving it. For the idealist reading not only lacks textual evidence in the dialogue and the related writings but also makes nearly incomprehensible Leibniz's own efforts in the metaphysics of motion and matter. Leibniz's writings of this period document an ongoing struggle to articulate a coherent account of the nature of the corporeal world as an infinitely divided plenum of moving beings. One puzzle after another emerges in Leibniz's inquiries, and his philosophical labors involve taking each as it comes and striving to find a solution. I think only a "realistic" interpretation can make natural sense of Leibniz's pursuits here. Consider, for example, the puzzle concerning the unity of corporeal beings one finds being explored in the texts of the 1680s. The state of play in his metaphysics after the *Pacidius* leaves Leibniz with a world of matter that is divided into parts within parts *ad infinitum*. In the years immediately following, he comes to worry anew about how there could be anything that is truly "one," any "true unity," if everything is always subdivided into finer parts (cf. A VI,4,1464., 1988). If nothing is truly *one*, then nothing *is* at all (G II,97, 251); hence there must be something in virtue of which the beings in the corporeal world *are* true unities despite their endless subdivision into parts. The metaphysical hypothesis to which Leibniz finds himself driven in 1679 in order to answer this question about the unity of corporeal beings in a world of infinitely divided matter involves the rehabilitation of the theory of substantial forms[47]—a theory which itself becomes the subject of intense philosophical scrutiny in his own writings for at least another decade and famously draws the attention of Leibniz's correspondents Arnauld and Fardella. An idealistic reading of the *Pacidius* metaphysics can make little sense of any of this, however. The puzzles about the true unity of divided corporeal beings that Leibniz is trying hard to solve should never arise from the world view of the idealistic reading of Leibniz—if there *are* no corporeal beings, there should be no *problem* concerning their unity, and thus no hypothesis of substantial forms demanded to solve it. The idealistic reading of the *Pacidius*-period metaphysics makes it all but impossible to understand Leibniz's inquiries or the content of the philosophy that is occupying his mind.

In contrast, a realistic reading of Leibniz's efforts to resolve the puzzles concerning the nature of motion and matter makes straightforward sense of his writings and paints a vivid picture of the world as he sees it. The realistic interpretation of Leibniz's views about motion and the corporeal world does not save his philosophy from the central difficulty that has been raised. It is thus a "problematic" reading of that philosophy; still, it strikes me as true to the philosophy itself. The presence of competing and irreconcilable conceptions of motion and reality in Leibniz's early thought means that if we are to regard his many ideas as yielding a "total metaphysical view," we must in the end also regard that view as collapsing when pressed to its limits. And I suggest that this is how we *should* regard those many ideas: they were, after all, developed in close proximity and in an astonishingly single-minded effort to resolve the paradoxes of the labyrinth of the composition of the continuum. But collapse of the "total metaphysical view" should not lead us to adopt a tragic attitude towards Leibniz on these points—as if to suppose that the presence of competing conceptions were a great flaw that undoes his philosophy. Quite the opposite is true. The tension between those conceptions is the very wellspring of Leibniz's best, most creative and most profound ideas. And only by continuing to pay close attention to this fact will we be able to come to a satisfactory understanding of Leibniz's first philosophy of motion.[48]

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[1]Charinus is the main subject of dialectical questioning for Pacidius, who explicitly plays the role of Socrates for the inquiry. (Indeed Pacidius is persuaded by the other characters to lead the inquiry in such a way as to demonstrate "the Socratic method of discussion as expressed in the Platonic dialogues". Cf. A VI,3,529-30.) It is sometimes said that Pacidius is *the* character who speaks for Leibniz. And certainly Pacidius seems to know his way through the material already (and moreover 'Pacidius' is a Latin play on Leibniz's own first name). But like Plato, Leibniz allows all the interlocutors collectively to make the case and to raise objections from various angles. Unlike the 1704 *New Essays*, for example, there is no single mouthpiece for the author's views in the *Pacidius*. In what follows the passages I shall be offering as Leibniz's own views are going to be taken not only from the lines spoken by Pacidius but also from those of Charinus and the others.

[2]I am responsible for translations of Leibniz's writings throughout this paper, but I have extensively consulted manuscripts of Richard Arthur's fine translation volume *Leibniz's Labyrinth: Writings on the Continuum Problem, 1672-1686*, forthcoming in the *Yale Leibniz Series*. For various passages I have consulted also Loemker 1969, Remnant and Bennett 1983, and Ariew and Garber 1989. Inaccuracies are my own. I abbreviate the primary texts thus: A = Berlin Academy Edition, *Samtliche Schriften und Briefe: Philosophische Schriften*; G = Gerhardt, ed., *Die Philosophischen Schriften* ; GM = Gerhardt, ed., *Mathematische Schriften von Gottfried Wilhelm Leibniz*. References to G and GM are to volume and page numbers; those to A are to series, volume and page.

[3]Leibniz's own rather esoteric views about death familiar from later writings (e.g., *Principles of Nature and Grace* §§ 6 and 12, *Monadology* §73) are not on display in the *Pacidius*; presumably, the example of life and death used here is offered only for its illustrative value.

[4]In an earlier draft Leibniz uses the more traditional phrase: *Tertium non datur*. It is unclear why he adopts the different formulation in the fair copy; perhaps it is just for a more natural phrasing in the dialogue.

[5]A series of elements is densely ordered if between any two elements x and y of the series there always lies a third element z also a member of that series.

[6]For a fine discussion of Aristotle's treatment see Michael J. White (1988). And for further discussion of these points in Leibniz's 1676 and 1671 writings, see [Author's #2] and White (1992). See also Basser (1998) and Beeley (1999).

[7]For good recent discussion of the Sorites, and defenses of the view that there must be such sharp boundaries between being poor and not poor (etc.), see Williamson (1993) and Roy Sorensen (*ms*.). For a discussion of Leibniz's views on the Sorites, and their development, see [Author's #4].

[8]For a full discussion of this use of the Sorites in the *Pacidius*, see [Author's #5]; for a related discussion, concerning Leibniz's use of Zeno's dichotomy paradox, see Richard Arthur, "Leibniz's Inversion of Zeno" in Mugnai (2001).

[9]White (1992) offers a similar reading of the passage about infinitesimal leaps, though he understands it to be invoking a somewhat stronger thesis, namely, that the principle of sufficient reason requires that there be *in rebus* a "foundational level of constituent parts" so that "a foundationless hierarchy of smaller and smaller orders of infinitesimals" would for Leibniz "constitute a 'vicious ontological regress'"(307).

[10]Leibniz's discussion in surrounding passages introduces a number of complications into his example, and the associated diagram is likewise complex. For the sake of clarity I have simplified the diagram here slightly, excerpting a single line from his figure appearing at A VI,3,547; and for continuity I have adapted Leibniz's labels, putting 'x' for his 'G'.

[11]Substituting 'x' for Leibniz's 'GH'; see the figure appearing at A VI,3,541.

[12]This definition of motion's being continuous repeats one from *Theoria Motus Abstracti* of 1671: "Motion is continuous ôr not interrupted by little intervals of rest"(A VI,2, 265).

[13]See [Author's #1] for a discussion.

[14]Richard Arthur has coined the *Pacidius* version of this argument "Leibniz's Diagonal Paradox." See Arthur (1999).

[15]For some discussion that addresses both the seventeenth-century techniques and Leibniz's later views about motion and dynamics, see Meli (1993) and Costabel (1973), and Garber (1982, 1992 and 1995). For a thorough account of the mathematical techniques of the period see Mancosu (1996) and Hofmann (1974).

[16]Notably "Règles du Mouvement dans la Recontre des Corps" in *Journal des Sçavants*, 18 March 1669. The first full publication of *De Motu* came only in 1703, eight years after Huygens's death; see his *Oeuvres Completès* 16:1-186.

[17]The limit of a sequence of polygonal curves need not be a non-differentiable curve; if the "pieces" of the curve can be taken as infinitesimals "in the limit," for example, the resulting figure may be treated as continuous and differentiable in the Leibnizian calculus. And Leibniz allows this in later writings as an acceptable way to proceed in calculating idealized accelerations. But his sharp rejection of infinitesimals—which is fresh in his mind in late 1676—precludes this from being a strictly correct metaphysical understanding of the interval of accelerated motion in reality.

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[18]At A VI,3,533, near the outset of the dialogue, Leibniz writes, "And so just as an outstanding philosopher of our time rightly said that Geometry is Mathematical Logic, so I will boldly declare that Phoronomy is Physical Logic."
[19]This is why Leibniz bothers to point out that we assume *space* and *time* to be uniform; presumably no one needs to be reminded that we also assume them to be *unaccelerated.*
[20]Continuity is one matter, possession of a tangent at every point is another. A function may be continuous everywhere but lack tangents at certain points and thus not be differentiable everywhere. In 1872 Weierstrass famously succeeded in defining a function that is continuous but has *no* tangents to it: it is continuous but *nowhere* differentiable.
[21]The lines omitted from the passage at that point are: "And the distinction between these two contiguous lines actually divided from each other, and the one continuous line, is clear: it is, as Aristotle has already noted, that the endpoints *B* and *D* in the two contiguous lines are different, while in the one continuous line they coincide, just as we also noted above."
[22]Leibniz goes on, "We have rejected the leaps discussed above. Consequently temporary rests cannot be inserted into any motion, otherwise we will necessarily end up with leaps," and then inserts, in parentheses, the exchange between Charinus and Pacidius concerning infinitesimal leaps at A VI,3,564-5, discussed above.
[23]See section 7 below.
[24]For discussion see Boolos (1971).
[25]For good discussion see White (1988) and (1992).
[26]His *The Fractal Geometry of Nature* (1983) is the *locus classicus* for the field.
[27]This can be misleading for in fact there is a somewhat loose cluster of several different sharply defined notions of fractal dimension; as many as *ten* distinct notions of dimension have been isolated. Hausdorff (or Hausdorff-Besicovitch) dimension is the most powerful in the sense that it is the most general, covering the widest spectrum of cases.
[28]Thus Hausdorff dimension and much of higher fractal mathematics (so to speak) will be excluded along with set theory. Still, it is an interesting question whether, or to what extent, definitions of fractal dimension stronger than scaling dimension might be reconciled with Leibniz's views about the continuum and the foundations of mathematics. Box dimension, for example, which is highly general and covers many fractal structures that scaling dimension does not, is defined as the limit of a series of ratios of logarithms: d = limit as $\varepsilon \to 0$ of $\log N(\varepsilon)/\log \varepsilon$, where $N(\varepsilon)$ is the least number of boxes of side ε needed to "cover" the structure whose dimension is being assigned. The standard analysis of limits uses a point-set definition of the continuum and thus it represents the continuum as an infinite collection of points—something contrary to Leibniz's own fundamental views. But the analysis of limit definitions is not the exclusive domain of contemporary set theory, and indeed other approaches to limits can be developed in other accounts of the continuum. Leibniz's own constructivist approach the continuum may then yield a constructivist counterpart to box dimension. See Breger (1990) and Lavine (1996) for some promising discussions of Leibniz's mathematics. White (1988) offers some reasons for thinking that a point-set approach to the continuum might not be irreconcilable with the Aristotelian idea that points are only bounds and not parts of continua (in standard point-set topology, "points" or singleton sets are never open sets for a continuum, hence not "parts" but only limits of it in an important sense); since Leibniz follows Aristotle on this matter, White's discussion will be instructive here as well.
[29]Scaling dimension is often used to guess the Hausdorff dimension of a given structure, and for cases of classic self-similar fractals this method is perfectly accurate.
[30]For complete presentations of fractal dimension, see Peitgen, Jürgens and Saupe (1992), and Edgar (1990). Compare also Courant, Robbins and Stewart (1996).
[31]*N.B.*: Size here is *scale size* rather than (say) area or volume: just as the scale size of a one-to-five map (e.g. "one inch equals five miles") is twice that of a one-to-ten map, a square with side 2*k* is twice the scale size of a square with side *k*.
[32]Von Koch (1904).
[33]Peitgen, Jürgens and Saupe (1992), 89.
[34]Mandelbrot (1983), and Bouquiaux (1994) note this connection, especially with respect to Leibniz's later writings such as the famous sections 62-64 of the *Monadology*; Deleuze (1993, p.16) also picks up on the idea of fractal structure to describe the "folding of matter" in Leibniz's metaphysics. None of those commentaries observes the source of the fractal conception in the early writings or its role in the *Pacidius* theory of motion.
[35]It exposed Leibniz to a famous raking from Jonathan Swift in verse:

So, naturalists observe, a Flea
Hath smaller fleas that on him prey,
And these have smaller Fleas to bit 'em,
And so proceed ad infinitum. ("On Poetry, a Rhapsody" (1733), lines 337-340)

This is meant as a lyrical *reductio* of the inference from the early observations of microscopists that there are "animalcules" living in tiny water droplets to the view that there exists an infinite and bottomless hierarchy of such creatures everywhere in nature. A century and a half later the mathematician Augustus DeMorgan would offer his own variation on Swift's lines:

Great fleas have little fleas
upon their backs to bite 'em,
And little fleas have lesser fleas,
and so ad infinitum,
And the great fleas themselves,
in turn, have greater fleas to go on,
While these again have greater still,
and greater still, and so on. (*A Budget of Paradoxes* (1872), Vol. 2, 191)

Yet the fleas-on-fleas image plays a slightly different role in DeMorgan's hands: there is no mention here of what the "naturalists observe," and the mathematical parallel with Leibnizian doctrine has been sharpened. DeMorgan goes on to suggest that it is an empirical matter whether the universe has such a structure. What began as satire is evolving into a new topic for mathematical inquiry.

[36]Among the various set problems in fractal mathematics that bear Leibniz's name, one example is that of filling a planar space with non-overlapping circles; it is called a *Leibniz packing* and was inspired by Leibniz's own discussion of circle-packing (cf. G II,306) and the three-dimensional case of sphere packing (cf. A VI,3,525, A VI,4,1399).

[37]Mandelbrot records Leibniz's discussion in the letter to de l'Hospital in "free translation" that omits the technical points but still offers a nice summary of Leibniz's remarks: "John Bernoulli seems to have told you of my having mentioned to him a marvelous analogy which makes it possible to say in a way that successive differentials are in geometric progression. One can ask what be a differential having as its exponent a fraction. You see that the result can be expressed by an infinite series. Although this seems removed from Geometry, which does not yet know of such fractional exponents, it appears that one day these paradoxes will yield useful consequences, since there is hardly a paradox without utility. Thoughts that mattered little in themselves may give occasion to more beautiful ones"(cf. GM II, 300-302). Leibniz's subsequent (and less florid) letter to Bernoulli of 28 December, 1695 (GM III.i,226-9), refers to the letter to l'Hospital and elaborates much the same technical material.

[38]Mandelbrot notes *In Euclidis □□□□□* for Leibniz's "scaling" definition of a straight line as "a curve any part of which is similar to the whole" at GM V,185 but appears not to have observed the reference to intermediate dimensions three pages later; Bouquiaux (1994) notes it at 241-2 fn.5.

[39]It is not essential to this problem that motion be *fractal* in its structure; it suffices that motion be actually infinitely subdivided everywhere—as Leibniz expressly maintains it to be (cf. A VI,3,565f.). The fractalist reading of the structure of the infinite division of motion just makes the difficulty especially vivid by highlighting the point that motion will be "all singularities."

[40]For a more extended account of this criticism, see [Author's #2].

[41]For a more detailed discussion, see [Author's #5].

[42]See Arthur (1989).

[43]Apart, that is, from the following hint from the dialogue's end as Pacidius, now recounting the long-past discussion to Alethophilus: "I added one more fruit of this demonstration, which was that from this it would appear that action is something very different from change, and that a thing can act without undergoing a reaction, a fact that is in turn of great utility in divinity, as everyone acknowledged with applause"(A VI,3,571).

[44]For some discussion, see Garber (1995) and Costabel (1973).

[45]For discussion, see [Author's #3]. Leibniz will, of course, eventually come to reject this essentially Cartesian idea; for the classic formulation of his argument, see article 13 of *De Ipsa Natura*.

[46]For some recent discussions concerning the status of idealism in Leibniz's later philosophy, see Garber (1985, 1995), Sleigh (1990), Adams (1994), Rutherford (1990, 1995), Hartz (1998), and Phemister (1999).

[47]Cf. A VI,4,1399f.; also cf. 1460f.,1465f., 1495, 1544f., and G IV,478f.

[48]My thanks to [...]

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