ACTUAL INFINITESIMALS IN LEIBNIZ'S EARLY THEORIES OF THE CONTINUUM[1]

by

Richard T. W. Arthur,

McMaster University

[1] It is a pleasure to acknowledge a sabbatical leave from my former institution, Middlebury College, during which the bulk of this paper was written.

INTRODUCTION

Gottfried Leibniz's views on the status of infinitesimals are very subtle, and have led commentators to a variety of different interpretations. There is no proper common consensus, although the following may serve as a summary of received opinion: Leibniz developed the infinitesimal calculus in 1675-76, but during the ensuing twenty years was content to refine its techniques and explore the richness of its applications in co-operation with Johann and Jakob Bernoulli, Pierre Varignon, de l'Hospital and others, without worrying about the ontic status of infinitesimals. Only after the criticisms of Bernard Nieuwentijt and Michel Rolle did he turn himself to the question of the foundations of the calculus and formulate his mature view that infinitesimals are mere fictions.[2] In many quarters, to boot, this mature view is seen as somewhat unfortunate, especially since the work of Abraham Robinson and others in recent years, which has succeeded in rehabilitating infinitesimals as actual, non-Archimedean entities.[3] A dissenting view has been given by Ishiguro, who argues (in my opinion, persuasively) that Leibniz's *syncategorematic interpretation* of infinitesimals as fictions is a conceptually rich, consistent, finitist theory, well motivated within his philosophy, and no mere last-ditch attempt to safeguard

---

[2] For a good statement of this position, see in particular D. M. Jesseph, "Leibniz on the Foundations of the Calculus: the Question of the Reality of Infinitesimal Magnitudes", *Perspectives on Science* 6: 1 & 2 (1998), pp. 6-40: "the fictional treatment of infinitesimals clearly appears designed in response to [Wallis and Bernoulli] and to the critics of the calculus [Nieuwentijt and Rolle]. If I am right we can see this doctrine take shape through the 1690s as Leibniz tries to settle on an interpretation of the calculus that can preserve the power of the new method while placing it on a satisfactory foundation." (p. 8) Cf. also Detlef Laugwitz: "It was not before 1701 that Leibniz was forced to clarify his opinions, both mathematically and philosophically, on the use and nature of infinitesimals": "Leibniz' Principle and Omega Calculus", pp. 144-154 in *Le Labyrinth du Continu*, ed J-M. Salanskis and H. Sinaceur, Paris, 1992, p. 145.

[3] But see the authoritative study by P. Ehrlich: "The rise of non-Archimedean Mathematics and the Roots of a Misconception : the Emergence of non-Archimedean *Grössensysteme*", forthcoming in *Archive for History of Exact Sciences*, for an explosion of the myth that non-Archimedean systems and infinitesimals were purged from mathematics by Cantor and Weierstraß until the advent of Non-Standard Analysis.

his theory from foundational criticism.[4] But Ishiguro doubts that Leibniz ever held infinitesimals to be actually infinitely small non-Archimedean magnitudes before developing this interpretation.

In this paper I attempt to throw light on these issues by exploring the evolution of Leibniz's early thought on the status of the infinitely small in relation to the continuum. The picture that emerges differs in one way or another from all those detailed in the previous paragraph. For one can distinguish among Leibniz's early attempts on the continuum problem three different theories involving infinitesimals interpreted as non-Archimedean magnitudes: (i) the continuum consists of assignable points separated by unassignable gaps (1669); (ii) the continuum is composed of an infinity of indivisible points, or parts smaller than any assignable, with no gaps between them (1670-71); (iii) a continuous line is composed of infinitely many infinitesimal lines, each of which is divisible and proportional to an element (*conatus*)of a generating motion at an instant (1672-75). By early 1676, however, he has already reached the conclusion that (iv) infinitesimals are fictitious entities, which may be used as *compendia loquendi* to abbreviate mathematical reasonings; they serve as a shorthand for the fact that finite variable quantities may be taken as small as desired, and so small that the resulting error falls within any preset margin of error. Thus on the reading I propose here, Leibniz arrived at his interpretation of infinitesimals as fictions already in 1676, and not in the 1690's in response to Nieuwentijt's and Rolle's criticisms, whatever may have been his later hesitations.

Let me begin with an overview of the argument of the paper.

---

[4] See H. Ishiguro: *Leibniz's Philosophy of Logic and Language*, (Second Edition), Cambridge, 1990.

The most important text for understanding Leibniz's theory of the continuum prior to 1676 is the *Theoria Motus Abstracti* (*TMA*), published in 1671. For it is in the section of this work titled *Fundamenta Praedemonstrabilia* ("Predemonstrable Foundations") that Leibniz gives his most explicit treatment of his early ideas; and in the same year Leibniz wrote a barrage of letters to leading intellectuals in Europe touting the theory of the continuum given there, stressing its claims to solve numerous problems that had previously been considered intractable, among them the problem of the original cohesion of corpuscles, and the mind-body problem. The tract begins with the claim that there are actually infinitely many actual parts in the continuum, and then proceeds to outline a novel theory of points. Although, frustratingly, Leibniz does not explicitly identify the parts with the indivisible points, this seems to be his intention, as I shall argue below. Leibniz has three main lines of justification for these points: the success of Cavalieri's method of indivisibles, for which they are supposed to provide a foundation; an argument for their existence based on an inversion of Zeno's dichotomy; and an appeal to the existence of horn angles as examples of actual infinitesimals.[5] These points or indivisibles are distinguished from minima, or partless points, which Leibniz rejects, and are characterized as lacking extension, but nonetheless containing parts having no distance from one another, what he calls "*indistan*t" parts.

From this foundation we can work both backwards and forwards. In *Fundamentum* (7) of the *TMA*, Leibniz writes "Motion is continuous, i.e. not interrupted by any little intervals of rest" (A VI, II, N41). The significance of this is at least partly biographical, since Leibniz himself had

[5] I am using the word 'infinitesimal' throughout this paper as a convenient substantive for the 'infinitely small', and as synonymous with Leibniz's "unassignably small". Leibniz did not begin referring to infinitely small things as infinitesimals until later in his career. Ironically, he attributed the term to Mercator when it was original with Wallis, of whose work Mercator had published an expansion *without* using Wallis' new term! For a thorough account and analysis of all this, see the fine study by E. Pasini: *Il reale e l'immaginario: La fondazione del calcolo infinitesimale nel pensiero di Leibniz*, Torino, 1993, esp. pp. 7-8, 19-24.

previously promoted such a theory. In the first section below I offer a reconstruction of Leibniz's early version of this theory, and speculate on his reasons for abandoning it.

Working forwards, there is an unresolved tension in the *TMA* between the rejection of minima and the promotion of indivisibles, notwithstanding the distinction Leibniz draws between them in the justifications he gives of indivisibles. This is the tension between the tacit claim that the continuum is composed of indivisibles and the claim that they are the beginnings of intervals, which is essential to one of the arguments justifying them. For if every indivisible is the beginning of a line, and lines are composed of them, then they are vulnerable to the same objections (given by Sextus Empiricus in antiquity) that Leibniz had already used against minima. Leibniz apparently realized this in late 1671, and in a piece written in late 1672 he attempts to rectify this difficulty by rejecting indivisibles for this reason, but retaining actual infinitesimals that are justified by their proportionality to incipient motions. This is the theory I analyze in the third section. In it infinitesimals are now homogeneous with the continuum they compose, and lines are no longer regarded as composed of points, which Leibniz now characterizes, following Aristotle and Ockham, as endpoints or modes having no autonomous existence. What we have here is in fact an actualist interpretation of the infinitesimal or differential calculus, with the actuality of its infinitesimals constrained by the requirement that they be founded in motion.

There is, however, an unresolved difficulty in this theory too. For the defence of Cavalieri's indivisibles (now re-interpreted à la Pascal as extended and divisible) depends on the idea that an actually infinite number of them can compose a finite extension, as in the *TMA*. But in 1672 Leibniz had already convinced himself that there is no actually infinite number or magnitude, if this is interpreted as a number or magnitude greater than all finite numbers or magnitudes. This

throws into question any interpretation of infinitesimals as inverses of the infinitely large, as are Leibniz's infinitesimals, whose ratios to finite quantities are "as 1 to ∞". Leibniz appears to have put his doubts on hold while developing the calculus, but in 1676 he finally comes to terms with such difficulties in a series of papers written prior to his taking up his appointment to the court in Hanover. At about the same time that he gives a rigorous justification of the use of infinitesimals and infinities in quadratures (the calculus), he also demonstrates that the endeavours or *conatus* of his early work are not after all infinitely small motions, but arbitrarily small finite ones. Thus Leibniz comes to reject the actuality of infinitesimals in favour of an interpretation of them as fictions. The paper concludes with a brief examination of this syncategorematic interpretation of infinitesimals, which I argue is implicit in his dialogue penned in November (NS) 1676, *Pacidius Philalethi*.

PHASE 1 (1669): UNASSIGNABLE GAPS IN THE CONTINUUM

(i) the continuum is composed of assignable points separated by unassignable gaps; in particular, the motion of a body consists in its creation at assignable instants, its being non-existent in between.

The textual basis for Leibniz's first theory of the continuum is slender, and has to be pieced together from unpublished fragments and his own later testimony. In the *Phoranomus*, a dialogue Leibniz penned while he was developing his dynamics in 1689, he testifies that in his youth he had held that "a slower motion is one interrupted by small intervals of rest"—a doctrine which had been propounded by Arriaga and promoted by Mersenne and Gassendi as the soundest way of solving the problem of how one continuous motion can be faster than another. This is borne out by a passage in Leibniz's unpublished *De rationibus motus* of 1669:

> "Whatever moves more slowly does so because of several little intervals of rest (*quietulas*) interspersed. What moves more quickly does so because of fewer. A little interval of rest is an existence in the same place for a time smaller than any given." (A VI, II, 171)

What is interesting here is Leibniz's characterization of an interval of rest as an existence for a time "smaller than any given", which he explicitly distinguishes from moments or instants in the rigorous sense of points or true minima of time. For Gassendi, by contrast, the duration of such a *quietula* would be the smallest time into which a given interval could be physically divided, a "physical point" of time, extremely small, insensible, but finite. The physical continuum is thus, for Gassendi, in fact a discontinuum, even if the discontinuity is not discernible by the senses. But for Leibniz "Things which collide remain in the same place for a time smaller than any given, yet for longer than a moment." (*ibid.*) A quantity smaller than any given is what Leibniz elsewhere calls an "unassignable" or infinitely small quantity.

Now this implicit mention of unassignable times tallies with the theory of motion that Leibniz describes with pride to his former teacher Jakob Thomasius in a letter of the same period:

> "Nothing must be supposed in bodies which does not flow from the definition of extension and antitypy. But only magnitude, figure, situation, number mobility, etc. flow from them, whereas motion itself does not. Hence, properly speaking, there is no motion in bodies existing as a real entity in them, but as I have demonstrated, whatever moves is continuously created, and bodies are something at any instant assignable in a motion, but are nothing at any intermediate time between the instants in an assignable motion—a view

unheard of till now, but one that is plainly necessary, and that will silence the voice of the atheist." (A II, I, 23-24; April 30th 1669)[6]

This is a somewhat puzzling passage, since Leibniz appears here to be claiming originality for a theory of continuous creation, when it would have been perfectly well known to both himself and Thomasius that Descartes (and many others, such as Erhard Weigel, Leibniz's teacher in Jena) had proposed such a view.[7] But I suggest that what is original about his proposal is not the idea of continuous creation itself, but this interpretation of motion as consisting in a body being created in a given location at an assignable time, and then lapsing into non-existence "at any intermediate time between the instants in assignable motion". That is, he conceives there to be times "*between* the instants in an assignable motion", although these will be unassignable. On this reading, Leibniz conceives of a body's motion to consist in its being created at each assignable instant, and to be non-existent for unassignable intervals between the instants. Thus creation will be continuous, in that there will be no assignable time at which the body is not being created at a different place; yet motion will be "metaphysically discontinuous" (to use Leibniz's own later term for this), in the sense that it does not continue beyond the instant at which it is created.

A possible objection to identifying the theories of motion given in the two passages is the discrepancy between the claim in the *De rationibus motus* that the body is "in existence" for an unassignable time between instants or moments, and the claim in the letter to Thomasius that a

---

[6] All translations in this paper are my own. Most are from *G. W. Leibniz: The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672-1686*, selected, translated, edited and with an introduction by Richard T. W. Arthur, New Haven and London, 2001; abbreviated *LLC*.

[7] Thus Leroy Loemker writes: "Leibniz's theory of 'continuous creation here seems merely to mean the source of all motion in God and is therefore very similar to the Cartesian opinion which he later criticized." (Loemker, Leroy, ed. *Gottfried Wilhelm Leibniz: Philosophical Papers and Letters*. 2nd ed. Dordrecht, 1976; p. 104).

moving body is "nothing" at such a time. But this discrepancy is softened when one takes into account Leibniz's belief that matter "is nothing if it is at rest". As he argued explicitly in another fragment from about this time, "Matter is nothing if it is at rest. There is a demonstration of this. For whatever is not sensed is nothing. But that in which there is no variety is not sensed." (A VI, II, $N42_3$).

Thus the picture we have is this: at any assignable instant, the body is at an assignable point in space. But these assignable instants are separated by unassignable times during which the body is at rest. There being nothing in such a body by which it can even be discriminated, it is "nothing" during these times between the instants at which God creates it.

Now I am not suggesting that we have here a sophisticated mathematical theory of motion. As Leibniz himself says, these are speculations "pleasing to an adolescent", dating from his fallow youth "when I was not yet versed in Geometry"[8]. Nevertheless, they are not necessarily therefore hopelessly naïve. In fact, they bear a remarkable resemblance to a theory of motion recently proposed by William McLaughlin and Sylvia Miller, which utilizes concepts of Edward Nelson's "Internal Set Theory" to resolve Zeno's paradoxes of motion.[9] On this interpretation, a (numerical) *infinitesimal* is defined as an entity "greater than 0 and less than every positive standard real" (376). That is, the real line is comprised of points corresponding to the reals, which Nelson calls "standard numbers", but there is an infinitesimal interval on each side of every such point. Nevertheless, no two standard numbers can differ by an infinitesimal interval. For the difference between any two standard numbers is standard: but an infinitesimal is by

---

[8] *Phoranomus*, *Dialogus II*; ed. André Robinet: in *Physis* 28, n. 23, 1991, 797-885; p. 803.

[9] W. I. McLaughlin and S. Miller: "An Epistemological Use of Nonstandard Analysis to Answer Zeno's Objections Against Motion", in *Synthese* 92, pp. 371-384, 1992.

definition less than any standard number. McLaughlin and Miller propose to solve Zeno's paradox of motion on this basis. As McLaughlin explains it in a companion article, their argument is that "a trajectory and its associated time interval are in fact densely packed with infinitesimal regions", so that, although no motion is taking place at any of the assignable instants that can be labeled by standard numbers, it can nevertheless occur in the infinitesimal regions on either side of them.[10]

Now it seems that this argument could apply equally to Leibniz's first theory. For here too no two assignable instants can be separated by an interval of time that is infinitesimal. For the difference between two assignable instants must be an assignable interval; but an infinitesimal interval is unassignable. This means that we cannot suppose the continuum to be *composed* of assignable instants separated by unassignable gaps, even if it consists of them. Thus there is a tension between the claim that the motion is continuous, in that there is no assignable instant at which it will not exist, and that it is discontinuous, in the sense that there really are gaps in the motion, even if unassignable, so that motion requires God's repeated acts of re-creation. Leibniz shows some awareness of this tension when he makes a last attempt to finesse it in his dialogue on the continuity of motion *Pacidius Philalethi*, composed in 1676:

> "And these kinds of spaces are taken in geometry to be points or null spaces, so that motion, although metaphysically interrupted by rests, will be geometrically continuous—just as a regular polygon of infinitely many sides cannot be taken metaphysically for a circle, even though it is taken for a circle in geometry, on account of the error being smaller than can be expressed by us." (*LLC*, n. 62, p. 409)

---

[10] W. I. McLaughlin: "Resolving Zeno's Paradoxes", in *Scientific American*, November 1994, pp. 84-89: p. 87, c1.

But by 1676, as we shall see, Leibniz has moved beyond this theory, which is only mentioned by him here for the sake of completeness. He cancels this passage and omits it from the fair copy, after commenting that it is "not at all to be defended, lest the reasonings of geometry or mechanics be subverted by metaphysical speculations" (*ibid.*).

In fact, Leibniz had already abandoned this account in favour of a continuist metaphysics of motion within a year of writing the letter to Thomasius quoted from above. For although he thought well enough of that letter to have it published with his edition of Nizolius in 1670, one of the few changes in the published version is precisely the excision of the above passage. After the clause "But only magnitude, figure, situation, number mobility, etc. flow from them" he simply replaces the rest of the passage with the parenthetical remark, "(Motion itself does not flow from them, whence it follows that bodies do not have motion except by means of incorporeals)".[11] This is a nod to the new theory he developed in 1670 under the influence of Hobbes, where the continuity of motion is construed in terms of Hobbesian *conatus* or endeavours, reconstrued as incorporeal "beginnings" of motion. This theory, elaborated by Leibniz in a series of working papers over the following year, was published in 1671 as the *Theoria Motus Abstracti* (*TMA*).

PHASE 2: INDIVISIBLE, UNEXTENDED POINTS

(ii) The continuum is composed of indivisibles, defined as parts smaller than any assignable, with no gaps between them. Indivisibles have indistant parts, but no extension, and stand in the ratio of 1 to ∞ with the continuum they compose. Having parts, they have magnitude, so that any two indivisibles of the same order and dimension have a finite ratio.

[11] "(*Motus ipse ex iis non fluit, unde nec corpora motum habent nisi ab incorporeis*.)" A II, I, N54, 443.

Let us then turn to the theory Leibniz outlines in the *Fundamenta Praedemonstrabilia* of his *TMA*. He begins this account with the claim that there are actually infinitely many parts in the continuum, in defiance of Aristotle, Descartes and White (who denied the actuality, the infinitude, and the parts, respectively):

> "(1) *There are actually parts in the continuum*, contrary to what the most acute Thomas White believes, and
>
> (2) *these are actually infinite*, for Descartes's "indefinite" is not in the thing, but the thinker." (A VI, II, 264; *LLC*, 339)

Leibniz then proceeds to articulate a novel theory of points. He rejects the standard Euclidean definition of a point as *cujus pars nulla est* ("that whose part is nothing", or "that which has no part"), as well as Hobbes's proffered improvement, "that whose part is not considered", in favour of a conception of it as something unextended, but which nevertheless has parts that are "indistant" from one another:

> "(3) *There is no minimum in space or body*, that is, nothing which has no magnitude or part...
>
> (4) *There are indivisibles ôr unextended things*, otherwise neither the beginning nor the end of a motion or body is intelligible...
>
> (5) *A point is not that which has no part*, nor that whose part is not considered; but that which *has no extension*, ôr whose parts are indistant, whose magnitude is inconsiderable, unassignable, is smaller than can be expressed by a ratio to another sensible magnitude unless the ratio is infinite, smaller than any ratio that can be given. But this is the basis of the *Cavalierian Method*, whereby its truth is evidently demonstrated, inasmuch as one

considers certain rudiments, so to speak, or beginnings, of lines and figures smaller than any that can be given." (A VI, II, 264-65; *LLC*, 339-40)

It seems clear that Leibniz intends the indivisibles or "beginnings" of a body or motion occurring in (4) to be identified with the unassignables of (5), since they are both described as infinitely small unextended things. An odd feature of this presentation, though, is that these indivisibles or unassignables are not explicitly identified with the "actually infinite parts" referred to in (1) and (2), i.e. they are not said to compose the continuum.[12] Nevertheless, this must be what Leibniz intends. For in his letters he describes the *TMA* as providing a solution to the problem of the composition of the continuum in terms of points one greater than another, which would make little sense if the points were not parts of the continuum.[13] This is confirmed by a closer inspection of the theory. For in (13) Leibniz refers to these indivisible points as parts of space, albeit parts smaller than any given part. Indeed they must be, in order for a moving body to be said to occupy a greater part of space at the point of contact than it would if at rest (this is crucial to his theory of cohesion):

"(13) *One point of a moving body in the time of its endeavour*, i.e. in a time smaller than can be given, *is in several* places or *points of space*, that is, it will fill a part of space greater than itself, or greater than it fills when it is at rest, or moving more slowly, or endeavouring in only one direction; yet this part of space is still unassignable, or consists in a point,

---

[12] One might —as Leibniz himself would later— hold that the continuum is divided into an infinity of finite parts, separated by indivisible points that are not parts.

[13] "But what do I anticipate being clarified by this [theory of points]? I believe the Labyrinth of the Continuum can scarcely be escaped in any other way" (to Henry Oldenburg, 11 March 1671; A II, I 90); "[the TMA] examines the reasons for abstract motions, and unfolds the wonderful nature of the continuum... so that as one endeavour is greater than another, so is one point greater than another, in which way I not only escaped from that whole labyrinth of the continuum, but also saved the Cavalierian geometry of indivisibles" (to Lambert van Velthuysen, May 1671; A II, I , 97).

although the ratio of a point of a body (or of the point it fills when at rest) to the point of space it fills when moving, is as an angle of contact to a rectilinear angle, or as a point to a line." (A VI, II, 265; *LLC*, 340-41)

It seems very probable that Leibniz was inspired to construct this theory by Hobbes's attempt to provide a sound philosophical foundation for Cavalieri's Method of Indivisibles.[14] For not only does Leibniz interpret Cavalieri's indivisibles similarly to Hobbes, but two other features of Hobbes's analysis are also to be found in his own: (i) a proposed redefinition of 'point' intended to replace Euclid's, which is considered defective; (ii) a justification of these arbitrarily small points in terms of "horn angles" (a horn angle is the angle of contact between a straight line and a curve, usually the arc of a circle).

Leibniz's theory is by no means just a version of Hobbes's, however. In the first place, he rejects Hobbes's definition of a point as a line "whose length is not considered" (more precisely, a body whose length, breadth and depth are not considered)[15], opting instead for an interpretation of points as actually infinitely small, in opposition to Hobbes's finitism. He interprets the horn angles as support for this position, in that one horn angle may be bigger than another while both are less than any rectilinear angle that can be assigned.[16] Interesting too in this connection is Leibniz's passing mention of the Scholastic Theory of Signs in *fundamentum* 18. This appears to

---

[14] See Jesseph, *Foundations*, for a detailed treatment of Leibniz's debt to Hobbes. For a succinct account of Cavalieri's method, see K. Andersen: "Cavalieri's Method of Indivisibles", in *Archive for History of Exact Sciences* 31, 4, 1985, pp. 291-367.

[15] All Hobbes's mathematical objects are bodies: a surface is a body whose depth is not considered, a line a surface whose breadth is not considered. See Hobbes: *De Corpore*, VIII, 12; excerpts in *LLC*, 559.

[16] That Leibniz was not mistaken in taking horn angles for actual infinitesimals is shown by an interesting article by S. K. Thomason: "Euclidean Infinitesimals", in *Pacific Philosophical Quarterly* **63**, 168-185. Thomason shows that one could construct a consistent theory of horn angles within Euclidean geometry, in which they would indeed count as non-Archimedean infinitesimals.

have emboldened him in his idea that points, though unextended, may nevertheless have a structure or situation of (unextended) parts. That is, the parts will have a situation even though they are "indistant" or "lack distance" from one another.

The importance of this property of points is that it enables Leibniz to evade some of the traditional objections to composing the continuum from points. In his *Parmenides* (138a) Plato had argued that a thing without parts cannot have a situation, and Aristotle had built on this argument in his *Physics* (231b), where he argued that indivisibles, being partless, cannot be joined. Similarly, Sextus Empiricus argued that if a line were composed of points one would not be able to divide it, since a point has no parts.[17] Again, if a line were composed of partless points or minima, there would be as many points in the diagonal as in the side of a given square, since they can be put into a 1-1 correspondence; but then there will be none in the line that is their difference, contrary to the initial supposition that every line is composed of points.

Leibniz addresses both of these objections by acknowledging that they apply to true minima, or partless points, in contradistinction to the points he has defined:

> "(3) *There is no minimum in space or body*, that is, there is nothing which has no magnitude or part. For such a thing has no situation, since whatever is situated somewhere can be touched by several things simultaneously that are not touching each other, and would thus have several faces; nor can a minimum be supposed without it following that the whole has as many minima as the part, which implies a contradiction." (A VI, II, 264; *LLC*, 339)

The first objection does not apply to his own points because these are asserted to have parts, albeit unextended ones, and thus a situation to one another, even though the parts are indistant.

---

[17] Sextus Empiricus, *Against the Physicists* I, 288.

Moreover, since magnitude of a quantity is defined as "the multiplicity [*multitudo*] of its parts", Leibniz's points (unlike Galileo's *parti non quante*) may have a magnitude. Because of this, he assumes, they avoid Sextus's objection too.

The theory of magnitude of these points is further clarified by (6) and (10);

"(6) The ratio of rest to motion is not that of a point to space, but that of nothing to one.

(10) Endeavour is to motion as a point is to space, i.e. as one to infinity, for it is the beginning and end of motion." (A VI, II, 265; *LLC*, 340-41)

That is, the ratio of a point to a line is 1 to ∞, not 0 to 1. Points are not "nothings", as Wallis termed them,[18] but are proportional to the motions generating them. Take, for instance, a line segment of finite magnitude F. This is composed of an infinity of parts, each smaller than any assignable, whose magnitude is therefore F/∞. Points of different magnitudes are generated by motions at different uniform speeds:

"(18) *One point is greater than another, one endeavour is greater than another, but one instant is equal to another*, whence time is expounded by a uniform motion in the same line, although its parts do not cease in an instant, but are indistant. In this they are like the angles at a point, which the Scholastics (whether following Euclid's example, I do not know) called *signs*, as there appear in them things that are simultaneous in time, but not simultaneous by nature, since one is the cause of the other." (A VI, II, 266; *LLC*, 341)

Thus if we take two points *p* and *q* that are the beginnings of two different lines described in time T by the unequal uniform motions (whose speeds are) M and N, they will be proportional to

[18] Again, see Jesseph, *Foundations*, for an illuminating treatment of the relationship of the views of Wallis and Leibniz on the infinitely small.

the endeavours that are the beginnings of these motions, M/∞ and N/∞, resp. Therefore even though they are infinitely small they will be in the ratio M:N, i.e. in the same ratio as their generating motions. An infinity of points of length MT/∞ will compose a line of length MT, just as an infinity of endeavours M/∞ will compose the motion M.

In this last respect, the composition of a continuous motion M from an infinity of endeavours M/∞, the theory contrasts with Leibniz's earlier theory of metaphysically discontinuous motion, as he implicitly observes:

> "(7) Motion is continuous, i.e. not interrupted by any little intervals of rest. For
>
> (8) once a thing comes to rest, it will always be at rest, unless a new cause of motion occurs." (A VI, II, 265; *LLC*, 340-41)

Finally, Leibniz justifies the existence of these endeavours or beginnings of motions with the following ingenious inversion of Zeno's dichotomy argument:[19]

> "(4) *There are indivisibles or unextended things*, otherwise neither the beginning nor the end of a motion or body is intelligible. This is the demonstration: any space, body, motion and time has a beginning and an end. Let that whose beginning is sought be represented by the line *ab*, whose midpoint is *c*, and let the midpoint of *ac* be *d*, that of *ad* be *e*, and so on. Let the beginning be sought to the left, on *a*'s side. I say that *ac* is not the beginning, since *dc* can be taken away from it without destroying the beginning; nor is *ad*, since *ed* can be taken away, and so on. Therefore nothing is a beginning from which something on the right can be taken away. But that from which nothing having extension can be taken away is unextended.

---

[19] For a detailed analysis of the this inversion of Zeno's dichotomy and its place in Leibniz's thought, see R. T. W. Arthur: "Leibniz's Inversion of Zeno's Dichotomy," forthcoming in *Corporeal Substances and the Labyrinth of the Continuum in Leibniz*, eds. M. Mugnai and E. Pasini. *Studia Leibnitiana*, suppl.

Therefore the beginning of a body, space, motion, or time (namely, a point, an endeavour, or an instant) is either nothing, which is absurd, or is unextended, which was to be demonstrated." (A VI, II, 264; *LLC*, 339)

In calling this an inversion of Zeno's dichotomy argument I mean that, while Zeno argued for the unreality of motion on the grounds that the motion could never begin, Leibniz takes the reality of motion for granted and uses the dichotomy argument to argue that the beginning must be unextended. Indeed, since this argument is applicable to any subinterval of the motion, it entails the stronger conclusion that any subinterval whatever must contain an unextended beginning. Given the proportionality of points to endeavours, this argument therefore provides a powerful justification for Leibniz's notion of extensionless points.

There is, however, a problem of consistency with this theory that has been pointed out by other commentators. For the assumption that a line is composed of points —even points like Leibniz's that have parts and magnitude, but no extension— is just as susceptible to Sextus Empiricus's objection as the assumption of *true minima* which Leibniz had rejected in the *TMA*. He appears to have realized this late in 1671,[20] but the argument for it is given explicitly in a paper written in the winter of 1672/3 (*De minimo et maximo*)[21], where he now identifies indivisibles with minima and rejects both. His version there of Sextus' argument (which I have elsewhere dubbed "Leibniz's Diagonal Argument") runs as follows:

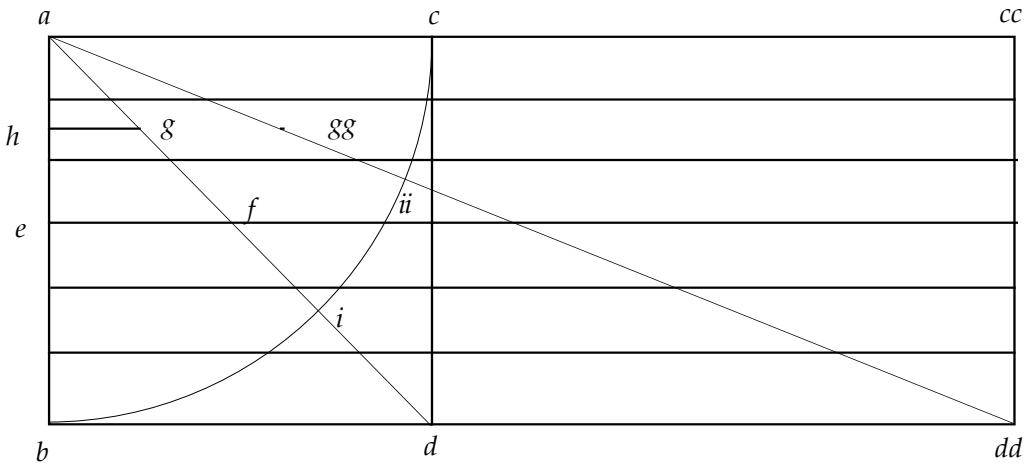
"*There is no minimum, ôr indivisible, in space and body.*

---

[20] In a letter to Arnauld dated November, 1671, Leibniz wrote: "there are no indivisibles, but there are unextended things" (A II, I, 172). P. Beeley takes this to have been Leibniz's position all along: see his *Kontinuität und Mechanismus*, esp. pp. 258-9. For criticisms, see O. B. Bassler: "The Leibnizian Continuum in 1671", in *Studia Leibnitiana* 30 (1998), no. 1, pp. 1-23; p. 19, and R. T. W. Arthur: "The Enigma of Leibniz's Atomism", in D. Garber and S. Nadler eds., *Oxford Studies in Early Modern Philosophy,* I (2003), pp. 183-227; 196.

[21] "*De minimo et maximo. De corporibus et mentibus* (*On Minimum and Maximum; on Bodies and Minds)*: A VI, III, N5; *LLC*, 8-19.

For if there is an indivisible in space or body, there will also be one in the line *ab*. If there is one in the line *ab*, there will be indivisibles in it everywhere. Moreover, every indivisible point can be understood as the indivisible boundary of a line. So let us understand infinitely many lines, parallel to each other and perpendicular to *ab*,



to be drawn from *ab* to *cd*. Now no point can be assigned in the transverse line or diagonal *ad* which does not fall on one of the infinitely many parallel lines extending perpendicularly from *ab*. For, if this is possible, let there exist some such point *g*: then a straight line *gh* may certainly be understood to be drawn from it perpendicular to *ab*. But this line *gh* must necessarily be one of all the parallels extending perpendicularly from *ab*. Therefore the point *g* falls—i.e. any assignable point will fall—on one of these lines. Moreover, the same point cannot fall on several parallel lines, nor can one parallel fall on several points. Therefore the line *ad* will have as many indivisible points as there are parallel lines extending from *ab*, i.e. as many as there are indivisible points in the line *ab*. Therefore there are as many indivisible points in *ad* as in *ab*. Let us assume in *ad* a line *ai* equal to *ab*. Now since there are as many points in *ai* as in *ab* (since they are equal), and as many in *ab* as in

ad, as has been shown, there will be as many indivisible points in *ai* as in *ad*. Therefore there will be no points in the difference between *ai* and *ad*, namely in *id*, which is absurd." (A VI, III, 97; *LLC*, 8-11)

From a modern perspective this argument is apt to seem fallacious: it looks as though Leibniz has conflated the measure of the set of points in a line with the number of points contained in it. Just because there is the same number of indivisible points in *ai* as in *ad*, it does not follow that their difference *id* has zero measure. But this presupposes a rather anachronistic point of view for appreciating this argument, for Leibniz's whole theory precisely depends on a notion of point as possessing a non-zero magnitude: this is what enables Leibniz to claim that one point may have a ratio to another. Also, prior to modern measure theory there was no way to compose a magnitude from points which lack magnitude.[22] Adopting a perspective that is more historically sensitive, one can treat Leibniz's argument on its own terms as follows. It can be seen to depend on four assumptions: (i) that there are points everywhere in a given line, each of which can be considered to be the beginning of any other line, and; (ii) that the given line can be regarded as composed of these points as parts; (iii) that all the points of any given line are of equal magnitude; and (iv) that the whole is greater than the part. Assumption (i) allows the establishment of a 1-1 correspondence between the points of any two lines, by connecting them with parallel straight lines. The trouble is that by (iii) each of the points on any one of the parallels connecting the lines *ab* and *ad* will be of equal magnitude, so that by (ii) the magnitudes of *ab* and *ad* will be equal. By a similar argument the magnitudes of *ab* and *ai* will be equal. Thus the magnitude of *ad*, the whole, will equal the magnitude of *ai*, the part, contradicting (iv).

---

[22] Cf. Spinoza, from his Letter on the Infinite: "For it is the same thing for a duration to be composed out of moments as for a number to arise solely by the addition of noughts (*Idem enim est durationem ex momentis, quam numerum ex sola nullitatum additione oriri*"; quoted from Leibniz's version, A VI, III, 280; *LLC*, 110-11).

Leibniz's solution is to give up his identification of the actually infinitely small with unextended points or indivisibles. That is, if the infinitely small "beginnings" in a line are taken to be indivisible in the sense of having zero extension, then there is nothing to prevent such points being taken as the endpoints of other lines, as in assumption (i). But this enables the Diagonal Paradox, as explained above. Consequently the idea of indivisibles or points of zero extension composing an extended line must be dropped. Leibniz's attempt to distinguish minima (having zero magnitude) from indivisibles (having zero extension) does not succeed.

Another way of expressing this point is in terms of dimensional homogeneity.[23] In characterizing his points as indivisible beginnings, Leibniz was trying to justify the idea of a point as a rudiment or beginning from which the line could be considered as generated. But the diagonal paradox throws into question the whole idea of the composition of the line from unextended points, and thus the composition of any continuum of dimension d from elements of dimension $d - 1$. The saving of Cavalieri requires the "points" to have an infinitely small extension, rather than to be unextended indivisibles. If points are considered as truly dimensionless or unextended, then the Diagonal Paradox shows that they cannot compose a line: their ratio to a finite line would be 0 to 1, not 1 to ∞, as intended. This realization leads Leibniz to modify his theory accordingly.

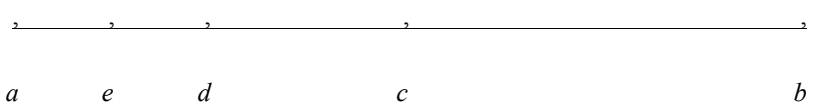
PHASE 3: INFINITELY SMALL LINES PROPORTIONAL TO ENDEAVOURS

(iii) a continuous line is composed of infinitely many infinitesimal lines, each of which is divisible and proportional to a generating motion at an instant (*conatus*) (1672-75).

[23] This point about dimensional homogeneity has been lucidly explained by Bassler in "The Leibnizian Continuum in 1671".

In *De minimo et maximo*, as we have seen, Leibniz uses the Diagonal Argument to reject indivisibles. But immediately afterwards he reaffirms the existence of infinitely small actuals or beginnings of motion with a reiteration of the Inverted Zeno argument:[24]

"*There are some things in the continuum that are infinitely small, that is, infinitely smaller than any given sensible thing.*



First I show this for the case of space as follows. Let there be a line *ab*, to be traversed by some motion. Since some beginning of motion is intelligible in that line, so also will be a beginning of the line traversed by this beginning of motion. Let this beginning of the line be *ac*. But it is evident that *dc* can be cut off from it without cutting off the beginning. And if *ad* is believed to be the beginning, from it again *ed* can be cut off without cutting off the beginning, and so on to infinity. For even if my hand is unable and my soul unwilling to pursue the division to infinity, it can nevertheless in general be understood at once that everything that can be cut off without cutting off the beginning does not involve the beginning. And since parts can be cut off to infinity (for the continuum, as others have demonstrated, is divisible to infinity), it follows that the beginning of the line, i.e. that which is traversed in the beginning of the motion, is infinitely small." (A VI, III, 98-99; *LLC*, 12-13)

This argument, as before, depends on an assumption (in contradiction to Zeno) that the phenomenon of motion is real, and (in agreement with Zeno) that, in order for there to be a real

[24] Although Leibniz appears to have already distinguished his points from indivisibles in his letter to Arnauld of 1671 (see note 19 above), here he goes further, characterizing the infinitely small not as unextended points, but as infinitely small lines.

motion, it must have an intelligible beginning. From this, however, a contradiction is derivable, if infinitesimals are thought of as preexisting parts of space and body:

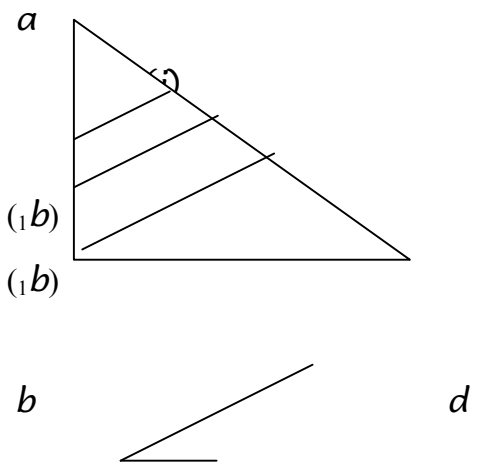
> "I shall show that if there is some space in the nature of things distinct from body, and if there is some body distinct from motion, then indivisibles must be admitted. But this is absurd, and contrary to what has been demonstrated. Suppose we understand a point as an infinitely small line, there being one such line greater than others, and this line is thought of as designated in a space or body; and suppose we seek the beginning of some body or of a certain space, i.e. its first part; and suppose also that anything from which we may cut off something without cutting off the beginning cannot be regarded as the beginning: with all this supposed, we shall necessarily arrive at indivisibles in space and body. For that line, however infinitely small it is, will not be the true beginning of body, since something can still be cut off from it, namely the difference between it and another infinitely small line that is still smaller; nor will this cease until it reaches a thing lacking a part, or one smaller than which cannot be imagined, which kind of thing has been shown to be impossible." (A VI, III, 99-100; *LLC*, 14-17)

This is a curious line of reasoning. Leibniz argues that if the infinitely small elements of a line are unextended or indivisible, as he had concluded in the *TMA*, then they would be susceptible to Sextus's refutation. If, on the other hand, they are regarded as infinitely small lines, then, so long as they are extended, they will not be true beginnings as required by the Inverted Zeno argument. Here he finds a third option. This is to regard them as infinitely small lines *modulo* a particular generating motion: infinitely small lines are contingent on, and must be defined in terms of, the infinitely small beginnings of motion or *conatûs* (ôr endeavour).

But if a body is understood as that which moves, then its beginning will be defined as an infinitely small line. For even if there exists another line smaller than it, the beginning of its motion can nonetheless be taken to be simply something that is greater than the beginning of some other slower motion. But the beginning of a body we define as the beginning of motion itself, i.e. endeavour, since otherwise the beginning of the body would turn out to be an indivisible. (A VI, III,100; *LLC*, 16-17)

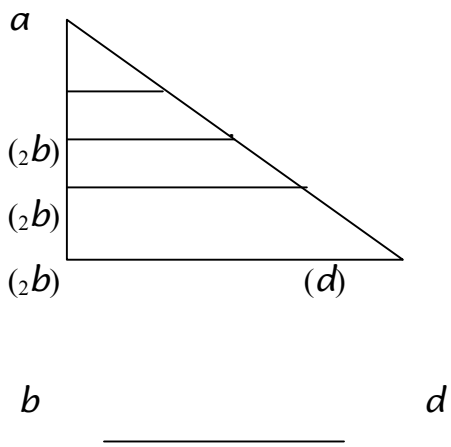
Figures

(i)



Regula (bi)

(ii)



Regula (bd)

θ

We can make sense of this as follows. Infinitely small lines are intelligible only in terms of their proportionality to the endeavours of corresponding generating motions. Thus if the

infinitesimals ($_1b$) and (i) in the lines *ab* and *ai* are generated by the motion of a *regula*[25] parallel to *bi* moving from *a* to *bi*, the infinitesimals in *ab* will be equal to those in *ai*; but they will be of a different magnitude than the infinitesimals ($_2b$) in *ab* generated by the motion of a *regula* parallel to *bd* moving from *a* to *bd*. In fact, if the *regula* (*bi*) moves with velocity v for a time t to reach *bi*, and the second *regula* reaches *bd* in the same time, the infinitesimals ($_1b$) and (i) will be of magnitude (vt/∞) and the infinitesimals ($_2b$) and (d) of magnitude ($vt\sec\theta/\infty$), since the latter will be generated by a motion whose effective velocity is $v\sec\theta$. But what this means is that infinitesimals exist as elements or actual parts of a line only relative to a given generating motion. But the same real line cannot really be composed of infinitesimals corresponding to different motions, as *ab* is in figures (i) and (ii). Yet the infinitesimals of lines from the two distinct motions can be compared.

Now the interpretation of Cavalierian points as indefinitely small lines is also in keeping with the interpretation Pascal gives them in his *Lettres de A. Dettonville, contenant quelques-unes de ses inventions de géometrie* (1659).[26] Thus when Leibniz reads the *Lettres* on Huygens' suggestion in the first half of 1673, he is already in a state of total receptivity to Pascal's reading. Actually, however, as Enrico Pasini has perceptively observed, Pascal does not interpret Cavalieri's indivisibles directly as infinitely small lines. Rather, he interprets indivisible points as marking the divisions of a line into indefinitely many such infinitesimal lines, and indivisible lines as dividing a plane into indefinitely small rectangles or parts. On this reading, the parts are in each case homogeneous with the continuum they compose, rather than being indivisible elements of one fewer dimensions. Pascal had written

---

[25] For the importance of the *regula* to Cavalieri's method, see Andersen: "Cavalieri's Method", pp. 299ff.

[26] This is argued in detail by E. Pasini: *Il reale e l'immaginario*, esp. pp. 50-59.

"Let there be understood to be an indefinite multiplicity of planes between them, parallel and equally distant (this means that the distance from the first to the second is equal to the distance from the second to the third, and to that from the third to the fourth, and so on), which planes cut all the proposed magnitudes into an indefinite multiplicity of parts, each one comprised between any two of these neighbouring planes." (Pascal, 1659, 7-8)

Pasini comments: "Such parts are, in distinction from the usual version of the method of indivisibles, comprised between the lines that individuate them, and not identical with them. They are therefore extended, and for this reason dimensionally homogeneous with everything of which they are a part."[27] This contrasts, for instance, with John Wallis (whom Leibniz had also just read at Huygens' suggestion), who had regarded Cavalieri's planes as directly composed from lines, which he allowed might be equated with parallelograms.[28] Thus Wallis's method fudges over a dimensional difference, and cannot be said to be either clear or rigorous. On Pascal's interpretation, on the other hand, as Pasini explains, whenever a surface is covered with lines that divide the area, they are understood to be distributed over the infinitely small parts of the straight line taken as the base of the figure, each of which functions as a unity, so as to generate equal rectangles of indefinitely small size:

"When one speaks of the *sum of an indefinite multiplicity of lines* one always has in view a certain line by the equal and indefinite parts of which they are multiplied. But when this line (by the equal portions of which they are understood to be multiplied) is not expressed, it is

---

[27] "*Tali parti sono, a differenza che nella abituale versione del metodo degli indivisibili, comprese tra linee che le individuano e non identiche con esse. Sono dunque estese e percìo omogenee per dimensione con il tutto di cui sono parte...*" Pasini, pp. 51-52, my translation.

[28] In his *Treatise on Conic Sections*, 1655, Prop. 1, Wallis had written of Cavalieri's planes as "composed of infinite parallel lines, or rather (as you may prefer) of infinite parallelograms of equal height, the height of each of which is therefore $1/\infty$ of the height of the whole". Cf. Pasini, pp. 45ff.

necessary to understand that it is that by whose division they originate [or by which they are multiplied]." (Pascal, 1659, 11; Pasini, p. 53)

This neatly resolves the difficulty of dimensional homogeneity. Each line (or ordinate) is multiplied by an infinitesimal segment of the line which functions as a unity (since the ratio of such successive equal parts is one), so that the area of the figure is composed from an indefinite multiplicity of indefinitely small areas. On this Leibniz follows Pascal:

"[I]n the Geometry of Indivisibles, when it is said that the sum of lines equals a certain surface or that the sum of surfaces equals a given solid, it is necessary for there to be given a unity, that is, for there to be a certain line to which they are understood to be applied, ôr into one of whose infinitely many equal parts, which represents the unity, they are multiplied, so that from them arise infinitely many surfaces, each of which is, however, smaller than any given surface." (Leibniz, Lh 35 15 1, f. 20; Pasini, p. 53)

Leibniz notes: "the *indivisibles* [of Cavalieri's *Geometria*] must be defined as infinitely small, ôr that whose ratio to a sensible quantity is infinity."[29] Similarly, in *On Minimum and Maximum* he had defined the "infinitely small things" in the continuum as "things infinitely smaller than any given sensible thing".[30]

A full account of this stage of Leibniz's thinking on infinitesimals would include a detailed description of his method of sums and differences. As is well known, he generalized results obtained with difference series involving discrete finite differences to the case of continuous

---

[29] *De admirandis arithmeticae infinitorum paradoxa* (On the Wonderful Paradox of the Arithmetic of the Infinite); Lh 35 15 1, f. 20v; first half of 1673; translated from the passage quoted in Pasini, p. 54.

[30] "*Infinite parva, seu infinities minora, quovis sensibili dato*"; A VI, III, 98; *LLC*, 12-13. The talk of "a more profound contemplation" also evokes Leibniz's boast in *De minimo* that "This wonderful method of demonstration, unnoticed by anyone else, became clear to me from a more intimate knowledge of indivisibles (*Mira et a nemine observata haec demonstrandi ratio mihi patuit, ex interiore indivisibilium cognitione*)"; A VI, III, 99; *LLC*, 14-15.

geometrical lines, which were regarded as composed of an infinity of infinitely small differences, or *differentia*. Thus given a series A, such as that of the reciprocal natural numbers $^1/_1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + {}^1/_6$ ..., and a second series B whose terms are the differences of consecutive terms of the original series, here $^1/_2 + {}^1/_6 + {}^1/_{12} + {}^1/_{20} + {}^1/_{30} + \ldots$, the sum of the B series of differences is the difference between the first and last terms of the original A series. Generalizing to infinitely small differences, the area under a curve B(x), consisting in the sum $\int$ of the infinitesimal elements B(x)dx from x = a to x = b, could be obtained analogously by taking the difference between the values of a second curve C(x), C(b) - C(a), where the curve C(x) is constructed so that value of B at x is the slope of C at x, $\{C(x + dx) - C(x)\}/dx$.

Different "progressions of the variables" would correspond to which infinitesimal was regarded as being held constant, acting as the "unit" multiplied into the ordinates to preserve their dimensional homogeneity. Thus, in contrast to Wallis's "arithmetic of the infinite", an area would not be composed of an infinity of lines, but of an infinity of infinitesimal rectangles, the ordinates B(x) of the derivative curve "applied to" (i.e. multiplied into) dx. But the elements B(x)dx are not elements in an absolute sense, since one could equally have taken the dy's as units.

Leibniz had thus come a long way from Cavalieri. But he retained the connection at the foundation of Cavalieri's method between the infinitesimals and the motions generating the figures. As he wrote to Malebranche in 1675, "it is necessary to maintain that the parts of the continuum exist only insofar as they are effectively determined by matter or motion" (Letter to Malebranche, March-April 1675 (?): G I, 322; Malebranche, *Oeuvres*, 97). The relativity of the composition of the continuum from infinitesimal parts to the progression of the variables

selected is still understood in terms of infinitesimals being defined by the endeavours of the corresponding generating motion.

By Spring of 1676, however, this situation has changed dramatically. In a paper written in early April, he refers to a "very recent demonstration" that endeavours are not, after all, infinitely small motions:

> "But on the other hand there is the great difficulty that endeavours are along tangents, so that motions will be too. For I have demonstrated elsewhere very recently that endeavours are true motions, not infinitely small ones." (A VI, III, 492; *LLC*, 75)[31]

This spells the demise of the actualist interpretation of infinitesimals of Leibniz's third theory. In a series of papers he strives to work out the significance of this for understanding the continuity of motion. But regarding infinitesimals themselves, from now on he regards them as useful fictions, without status as actual parts of the continuum.

CONCLUSION

In this paper I have tried to document the changes in Leibniz's understanding of the infinitely small in his early work. What we find there is surprisingly rich and varied. Leibniz appears to have entertained in succession several significantly different theories of the infinitely small, from the one implicit in his original conception of continuous creation and motion in 1670 through to his discretist re-interpretation of that theory in 1676. In between he had developed a continuist and non-Archimedean theory, based on Hobbes's endeavours and Cavalieri's indivisibles, involving points lacking extension, and then a second interpretation of Cavalieri that made the infinitely small extended and homogeneous to the continuum they compose, but made

[31] Exactly what demonstration Leibniz is referring to here is unclear.

their existence relative to a given motion. And during the same creative period in Paris, Leibniz not only developed the infinitesimal calculus, but justified it using the Archimedean Axiom. In the process he provided a new interpretation of infinitesimals as fictions, standing for the fact that finite variable quantities may be taken as small as desired, and so small that the resulting error falls within any preset margin of error. And we have seen that throughout this development, Leibniz's brilliant mathematical innovations are in constant interplay with his thoughts on natural philosophy and its metaphysical foundations.