The remarkable fecundity of Leibniz's work on infinite series

Richard T. W. Arthur

a review article on

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These two volumes of Leibniz's works, edited by Leibniz-Archiv der Niedersächsischen Landesbibliothek, Hannover, and published by the Berlin-Brandenburgischen Akademie der Wissenschaften and the Akademie der Wissenschaften in Göttingen in 2003, are the latest volumes of Leibniz's collected works, and maintain the usual exemplary standards we have come to expect of the Akademie edition. Series vii, volume 3 (hereafter A vii.3) contains Leibniz's studies, sketches and drafts on infinite series, sequences and differences written during his Paris years, 1672-1676. The second volume, series III, volume 5 (hereafter A iii.5) contains Leibniz's correspondence on mathematical, scientific and technical matters over the period from 1691 to 1693, as well as a supplement containing two letters from 1674-76.

As is well known, one of Leibniz's seminal insights in his work on series concerned sums of differences. If from a given series A one forms a difference series B whose terms are the differences of the successive terms of A, the sum of the terms in the B series is simply the difference between the last and first terms of the original series: "the sum of the differences is the difference between the first term and the last" (A vii.3, p. 95). This insight, so far as I now, has never been named; I shall call it the Difference Principle. Suitably generalized, it becomes the basis for the fundamental theorem of the calculus: the sum (integral) of the differentials equals the difference of the sums (the definite integral evaluated between last and first terms), $\int Bdx = [A]^f_i$. As is well known, the Difference Principle has its origin in the problem set him by Huygens in September 1672 to find the sum of the reciprocal triangular numbers. This date is, incidentally, confirmed by Leibniz himself in *Summa fractionum a figuratis, per aequationes* (A vii.3, p. 365), as well as by the first piece in A vii.3, *De summa numerorum triangulorum recipricorum* (p. 3), although in his *Origo inventionis trianguli harmonici* of the Winter of 1675-76 he misremembers it as "Anno 1673" (p. 712). Leibniz explains the algorithm in a letter to Meissner 21 years later: "If one wants to add, for example, the first five fractions [sc. reciprocal triangular numbers] from $^{1}/_{1}$ to $^{1}/_{15}$ inclusive, one takes the number of fractions, that is, 5, adds to 1 to get 6, makes from this the fraction $^{5}/_{6}$, which doubled is $^{10}/_{6}$, or $^{5}/_{3}$, which is as many as $^{1}/_{1} + ^{1}/_{3} + ^{1}/_{6} + ^{1}/_{10} + ^{1}/_{15}$, the same as if one had added these fractions together." (A iii.5, p. 581) That is, if the terms of series A are the reciprocal natural numbers, $^{1}/_{1}$, $^{1}/_{2}$, $^{1}/_{3}$, $^{1}/_{4}$, $^{1}/_{5}$, $^{1}/_{6}$, etc., the differences of the successive terms of the series B are $^{1}/_{2}$, $^{1}/_{6}$, $^{1}/_{12}$, $^{1}/_{20}$, $^{1}/_{30}$, etc., i.e. half of $^{1}/_{1}$, $^{1}/_{3}$, $^{1}/_{6}$, $^{1}/_{10}$, $^{1}/_{15}$, etc., (call this series C), so that the successive reciprocal triangular numbers are simply twice the differences of the successive reciprocal natural numbers. Thus, according to the Difference Principle, the first five reciprocal triangular numbers sum to twice ($1 - ^{1}/_{6}$), or $^{5}/_{3}$. That is, in general $\sum_{i=1}^{n} C_i = 2(A_1 - A_{n+1})$

Leibniz immediately extended this procedure to infinite series: suppose the sum $\sum_{i=1}^{\infty} A_i = ^{1}/_{1} + ^{1}/_{2} + ^{1}/_{3} + ^{1}/_{4} + ^{1}/_{5} + \dots$ is S(A), that of the reciprocal triangular numbers is S(C) and that of their halves $^{1}/_{2} + ^{1}/_{6} + ^{1}/_{12} + ^{1}/_{20} + ^{1}/_{30} \dots$ is S(B), so that S(C) = 2 S(B). Now subtracting B term by term from A gives $^{1}/_{2} + ^{1}/_{3} + ^{1}/_{4} + ^{1}/_{5} + \dots$, which is S(A) − 1. Thus S(A) − S(B) = S(A) − 1, giving S(B) = 1, and therefore S(C) = 2.

Clearly such a calculation does not meet modern standards of rigour, since A is a diverging series, so that the sum S(A) is not defined. Leibniz seems to have been sensitive to such a potential objection, since he allows that "this ought to be demonstrated to come out in the infinite" (362). He does this by taking an arbitrary y^{th} term as the *terminatio* of the series, where "*y* signifies any number whatever". For the reciprocal numbers the *terminatio* of this arbitrary finite series A(*y*) will be $1/y$, and for the series of reciprocal triangular numbers C(*y*), $2/(y^2 + y)$, since the y^{th} triangular number is $(y^2 + y)/2$. Thus when the series $B(y) = ^{1}/_{2}C(y)$ is subtracted from A(*y*), the *terminatio* of the resulting series will be $1/y - 1/(y^2 + y)$, or $(y^2 + y - y)/(y^3 + y^2)$, or $1/(y + 1)$. But this is the *terminatio* for A(*y* + 1). Leibniz does not complete this demonstration in these terms, preferring to proceed to a geometrical depiction in terms of a triangle, but it entails that for arbitrarily large *y*, the sum of the series B, which = $^{1}/_{2}$ the sum of the series C, is 1 − 1/(y + 1). So the sum of the reciprocal triangular numbers, C, approaches 2 arbitrarily closely as y is taken arbitrarily large.

Connected with this is the question of the area under the hyperbola. In N. 38_{10} Leibniz calculates that the area under the curve between x = 0 and x = 1, ACBEM on his diagram, will be $1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + {}^1/_6 \ldots$, that is, S(A). Under the same conditions, the finite space under the curve CFGB will be $1 - {}^1/_2 + {}^1/_3 - {}^1/_4 + {}^1/_5 - {}^1/_6 \ldots$ But subtracting the finite from the infinite space gives ${}^2/_2 + {}^2/_4 + {}^2/_6 \ldots$, that is, $1 + {}^1/_2 + {}^1/_3 + \ldots$, i.e. S(A). "Which is remarkable enough, and shows that the sum of the series $1 + {}^1/_2 + {}^1/_3$ + etc. is infinite, and therefore that the area of the space ACGBM, even when the finite space CBGF is taken away from it, remains the same, that is, it takes away nothing that can be denoted (*nihil notabile*). By this argument it can be concluded that the infinite is not a whole, but a fiction, since otherwise the part would be equal to the whole." (p. 468)

Here we can see prefigured Leibniz's mature doctrine of the infinite: "there is no infinite number, nor infinite line or other infinite quantity, if these are taken to be genuine wholes" (*New Essays*, §157). There is an actual infinite, but it must be understood syncategorematically: for a series to be infinite means for there to be "more terms than one can specify". Thus, since there is no completion of an infinite series, there is no infinite collection of its terms, and therefore, strictly speaking, no sum. But as the above calculation shows, one can still treat such an infinite series as approaching arbitrarily close to a quantity which can therefore be regarded as its sum, under the fiction that it has a last term or *terminatio*. The fictional infinitely small last term then stands for the fact that one can make the error in terminating the series after a finite number of terms smaller than any pre-assigned error.

This understanding of the sum an infinite series is made explicit by Leibniz in another unpublished paper of April 1676: "Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like." (A vi.3, 503). This is a good anticipation of the modern definition in terms of partial sums. But equally, it is a good example of how Leibniz had, contrary to the views of many, thought hard about the foundations of the infinite and infinitely small at the very time that he was formulating the infinitesimal calculus, and not only after it had been publicly criticized by Nieuwentijt and Rolle at the end of the seventeenth century.

For the fictional nature of infinitesimals is implicit in this doctrine of the syncategorematic infinite: the difference between, say, the dichotomy series $^1/_2 + ^1/_4 + ^1/_8 + ^1/_{16} + ^1/_{32} \ldots$ and its limiting sum of 1 is infinitesimal, smaller than any finite quantity; but this infinitieth term simply stands for an error that can be made as small as desired, while the sum is calculated under the fictional assumption that the limit is included in the series as *terminatio*.

There are other mathematical treasures in volume A vii.3. As the editors point out, one of the most important results of his extensive exercises on divergent harmonic series (in nos. 2, 8-10, 12, 14, 22, 27-30, 38, 47, 49, 52-55, 57, 68, and 71) is his discovery of the Harmonic Triangle, the analogue of Pascal's Arithmetic Triangle. This discovery is described in nos. 30 and 53, and discussed with Tschirnhaus in nos. 49 and 55. Another fruit of the Difference Principle is the link Leibniz finds in his researches on the reciprocal square numbers between the summation of the series $\sum_{n=2}^{\infty} 1/(n^2 - 1)$ and the sums of the difference series of the even and odd numbers in no. 15 (*De summa quadratorum in fractionibus*, March-May 1673). Then there are Leibniz's attempts to find analytic expressions for the circumference and area of the circle, centered around the series named after him which he discovered in the Fall of 1673: "It has been shown by me that given the ratio of an ongoing sum of harmonics in progression, produced to infinity, the quadrature of the circle can be had… For here it is, the series of fractions : $^1/_3 - ^1/_5 + ^1/_7 - ^1/_9 + ^1/_{11}$ etc." (271-2). And more recognizably, "If the diameter is 1, the circumference is 4/1 – 4/3 + 4/5 – 4/7 etc.", i.e. π , so that π/4, the quadrant of the circle, is "1 $- {}^{1}/_{3} + {}^{1}/_{5} - {}^{1}/_{7} + {}^{1}/_{9} - {}^{1}/_{11}$ etc." (469).

From the above examples one can appreciate just how important to his philosophical development was the mathematical work Leibniz did in Paris. For here we can not only see the seeds of Leibniz's mature doctrine of the syncategorematic but actual infinite and his characterization of infinitesimals as fictions, but connections with his metaphysics too. For example, these views are linked with the doctrine of the phenomenality of bodies. Leibniz regarded it as established that every body is infinitely divided into ever smaller finite parts by the incessant motions within it. As a consequence, every body is an infinite aggregate of these parts, with the infinite understood syncategorematically: there is no part so small that it does not contain another smaller part. But it follows from this doctrine of the infinite that a body cannot be regarded as a determinate whole. It appears as a whole to the senses; yet it is not truly one. But if there is no such thing as one body, "it follows that there are no bodies either, these being nothing but one body after another. Hence it follows that either bodies are mere phenomena, and not real entities, or that there is something other than extension in bodies." (A vi. 4, p. 1464). This was one of Leibniz's favourite arguments for his monads: "*I hold that where there are only entities by aggregation, there will not be any real entities.* For every entity by aggregation presupposes entities endowed with a true unity" (to Arnauld, 30th April 1687).

Again, there is a connection between his doctrine of the infinite and his rejection of atoms. This is seen in the exchange with Huygens in the second volume under review, A iii.5. There Huygens explains (letter of July 1692, N. 90, p. 340) that in rejecting "the Cartesian dogma that the essence of bodies consists solely in their extension," he finds it necessary to give bodies impenetrability or resistance to being broken apart, and to suppose this resistance infinite. "The hypothesis of infinite firmness seems to me very necessary, and I do not understand why you find it so strange, or how anyone could infer a perpetual miracle." To this Leibniz replies that when Huygens judges that his primitive firmness "has to be supposed infinite, for there is no reason to suppose it to be of some certain degree", it would indeed be arbitrary for every body to have the same finite degree of firmness. But "there is no absurdity in giving different degrees of firmness to different bodies, otherwise by the same argument, one could prove that bodies must have either infinite or zero speed" (iii.5, p. 392); but infinite speed is an impossibility on the Leibnizian view; rather, all it could mean would be that there was no upper limit to the finite speed a body could have. Similarly, the supposition of infinite firmness is unnecessary; one needs only to suppose that each body is composed of other bodies each with its own degree of firmness, and so on down: "there is no need to avail oneself of an original source of firmness at the start, any more than to avail oneself of atoms; it suffices for there to be conserved small bodies each of which already has its own firmness but which remain attached one to another, a little like two tablets which touch by their flat united surfaces, which the pressure of the surrounding matter prevents from separating all at once" (394). Again, infinite hardness is incompatible with the elasticity embodied in the Laws of Collision, since "the force of two equal atoms colliding head-on with an equal speed would be lost, since it is only elasticity (*resort*) which makes bodies rebound" (393). The perpetual miracle, on the other hand, stems from the assumption he attributes to Huygens (which the latter denies in a footnote) that "it is only mutual contact (*attouchement*) that acts as a glue"; for then, "since there is no natural connection between mutual contact and attachment, it is necessary for adhesion to follow from mutual contact, which happens by a perpetual miracle" (393).

Historians of mathematics will also find much to treasure in the correspondence in this second volume, A iii.5, as the period 1691-93 was one of the most productive phases of Leibniz's mathematics in his time in Hanover. Here one will find classic exchanges between Leibniz and Huygens and van Bodenhausen concerning the

Bernoullian Funicular Problem—that of finding the curve along which will lie a chain suspended from points of equal height—(nos. 7, 9, 13, 21, 24, 33, 34, 36 ,37), and with L'Hospital and Huygens on the so-called Florentine and Bernoulli Problems (nos. 138, 185, 191).

The correspondence of this period is also of particular relevance to Leibniz's dynamics, since he was still responding to the reception of his *Dynamica de potentia et legibus naturae corporeae* (1690), and wrote his *Essay de dynamique* in 1692. Here there is extended correspondence on matters dynamical with von Bodenhausen (nos. 25, 49 ,55, 171), L'Hospital (nos. 161, 173), Haes (nos. 56, 58) and Papin (nos. 57, 61, 72, 73, 76, 89, 95-97), of which Leibniz's letter to Papin of February 1692 (no. 61, pp. 256-26) is especially instructive. There is also much on fluid dynamics, stemming from the controversy between Guglielmini and Papin on the escape velocity of fluid flowing out of a pipe. In point of fact, as the editors point out, the correspondence with Papin, Haes, Huygens, Crafft and von Bodenhausen alone takes up fully one half of the volume. But of especial importance to Leibniz was his correspondence with Pfautz (no. 7), Johann Bernoulli (no. 202), Ramazzini (nos. 20, 51, 62, 67), Guglielmini (nos. 50, 66, 77), L'Hospital, Volckhamer (nos. 30, 35, 38, 44), Tschirnhaus (124, 130, 152, 165) and Newton (139, 194).

Particularly fascinating in a scholarly edition such as this to read all the marginal comments made by the recipients of the letters. Thus when Leibniz writes in his letter to Newton that he has no doubt that Huygens would have acknowledged the dignity and truth of Newton's discovery about gravity in the appendix of his *Traité de la lumière*, his remark is met by Newton's sardonic one-worder: "doubtful" (p. 513); while Huygens himself, on receiving a letter from Leibniz written the same week, responds testily to the latter's claim that his (Huygens') cubic atoms attach to one another to become new atoms, "They do *not* do so (*ils ne s'attachent pas pour devenir atomes*)", and to Leibniz's claim that "you had promised to explain one day how an inflexible body can rebound", "Prove it! (*prouvez*)", and "see our letters on this (*voir nos letters sur ceci*)" (p. 519).

Further physical topics covered in this correspondence are vortex theory, the shape of the earth, motion in resisting media. Aside from physics and mathematics, there are also interesting exchanges on medicine, biography, administration and economics, and technology.

Volume A vii.3 was edited, transcribed and dated by Eberhard Knobloch with Siegmund Probst as co-editor, Probst having succeeded Nora Gädeke as co-editor for the bulk of the manuscripts in April 1995. Knobloch and Probst provide a thorough and helpful introduction, detailing Leibniz's sources, explaining his terminology, and discussing the main points of interest. It will be essential for all scholars working on the history of mathematics in the seventeenth century.

Volume A iii.5 contains 203 letters, of which 137 were written either to or for Leibniz, and 66 by him. 140 of the texts have been published before, either in whole or in part. Many of these have appeared earlier in Gerhardt's mathematical volumes, or in the *Œuvres* of Denis Papin or Christiaan Huygens, or in the works of Newton and Tschirnhaus. The bulk of these letters are edited by Heinz-Jürgen Hess, with James O'Hara taking on the correspondence with Fatio, Guglielmini, Haes, Halley, Heyn, Huygens, Lauterbach, Ramzzini, J. J. Spener, Stark, Volckamer, Wachsmuth and Weigel. Their 58-page introduction (in German) is erudite and informative, and they provide a number of useful indices: a list and brief account of each of Leibniz's correspondents, and biographical, bibliographical and subject indices.