Leibniz's Archimedean infinitesimals

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ABSTRACT

Leibniz's theory of infinitesimals has often been charged with inconsistency. For example, characteristic formulas such as '$x + dx = x$' appear to be in straightforward violation of the law of identity. In this paper I offer a defence of Leibniz's interpretation of infinitesimals as fictions, arguing that with it Leibniz provides a sound foundation for his differential and integral calculus. In contrast to some recent theories of infinitesimals as non-Archimedean entities, Leibniz's interpretation was fully in accord with the Archimedean Axiom: infinitesimals are fictions, whose treatment as entities incomparably smaller than finite quantities is justifiable wholly in terms of variable finite quantities that can be taken as small as desired, i.e. syncategorematically. In this paper I explain this syncategorematic interpretation, showing how Leibniz used it to justify the calculus, as well as exploring its wider historical context and philosophical implications.

1. INTRODUCTION

It is a standard objection to the use of infinitesimals in calculus that their proponents want to have it both ways: on the one hand, since infinitesimals are not equal to zero, they can, unlike null quantities, have a finite ratio; but on the other hand, being incomparably small relative to a finite quantity, they can be taken to be equal to zero.[1] This was how George Berkeley objected to Newton's "moments", which Newton denoted by the letter 'o': "I admit that ... in the original notation x + o, o might have signified either an increment or nothing. But then which of these soever you make it signify, you must argue consistently with such its signification, and not proceed upon a double meaning." (Berkeley 1734, 74-75). John Earman (1975, 244) makes the same point about Leibniz's differences *dx*. He gives the example of Leibniz's derivation of the tangent to the curve

$$y = x^2/a, \tag{1}$$

in which Leibniz proceeds by considering the point ($y + dy$, $x + dx$) infinitesimally close to the point (y, x), thus arriving at the formula

$$dy/dx = (2x + dx)/a \tag{2}$$

Here the ratio dy/dx is defined because neither dy nor dx is strictly zero. Now Leibniz sets dx/a equal to zero on the grounds that, relative to the finite quantity x, dx is "incomparably small". This gives

$$dy/dx = 2x/a. \tag{3}$$

But now this equation must be interpreted as an exact equality, not as an "equality up to an infinitesimal quantity" (Earman, p. 245). To put the matter succinctly, a Leibnizian "equation" such as

$$x + dx = x \tag{4}$$

taken as a literal equation, has only the solution

$$dx = 0 \tag{5}$$

This justifies the move from (2) to (3), but is inconsistent with the interpretation of (2), in which neither dy nor dx is strictly zero. It is true that the error in equating (2) with (3) can be

[1] Cf. the discussion of John Earman (1975), p. 244: When $\Delta x = 0$ the slope of the tangent, $\Delta y/\Delta x$, becomes equal to 0/0, an unacceptable result. "Infinitesimals were seen as a way out. Being different from 0, they allow one to avoid the 0/0 ratio, but being incomparably small, they can be neglected."

"made smaller than any assignable", as Leibniz asserts, but, Earman insists, "a contradiction is a contradiction even if it is only a small one" (245).[2]

Gilles Deleuze gives a more subtle interpretation, coining the term "vice-diction" to describe such vanishingly small contradictions (1994, 46). He argues that "difference implies the negative, and allows itself to lead to contradiction only to the extent that its subordination to the identical is maintained" (xix). He correctly sees that Leibniz justifies his treatment of differentials by reference to his Law of Continuity. According to that principle, a parabola can be regarded as the limiting case of an ellipse as the second focus is removed from the first, or rest the limiting case of motion as it becomes gradually slower, or a point as the limit of a line as the line is reduced in magnitude, even though in each example the series of cases or species (ellipses, motions, lines) terminates in "an opposing quasi-species" (46), (parabola, rest, point).[3] "Likewise equality can be considered as an infinitely small inequality" (Loemker 1969, 352). "And although these terminations are excluded," Leibniz explains, "that is, are not included in any rigorous sense in the variables which they limit, they nevertheless have the same properties as if they were included in the series, in accordance with the language of infinites and infinitesimals." (Loemker 1969, 546) The neologism "vice-diction" is intended to capture this state of affairs. Although the parabola is essentially different from the ellipse, excluding it "*in essence*", it includes it as a limiting *case*: "It does not contain the other in essence, but only with respect to properties, in *cases*" (Deleuze 1994, 46). Thus "in the infinitely small, ... the unequal vice-dicts the equal, and vice-dicts itself, to the extent that it includes in the case what it excludes in essence" (46).

But how does this resolve anything? Is there still a contradiction at the level of properties or cases? "In reality," Deleuze admits, "the expression 'infinitely small difference' does indeed indicate that the difference vanishes so far as intuition is concerned." (46) "It matters little whether the supposed negative of difference is understood as a vice-dicting limitation or a contradicting limitation [as in Hegel]" (50). What is at issue, according to Deleuze, is the alternative between infinite and finite representation (176):

> That is why the metaphysical question was announced from the outset: why is it that, from a technical point of view, the differentials are negligible and must disappear in the result? It is obvious that to invoke here the infinitely small, and the infinitely small

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3 "a given ellipse approaches a parabola as much as is wished, so that the difference between" Loemker p. 352)

magnitude of the error (if there is 'error'), is completely lacking in sense and prejudges infinite representation. The rigorous response was given by Carnot in his famous *Reflections on the Metaphysics of the Infinitesimal Calculus*, but precisely from the point of view of a finite interpretation ... (177)

Thus Deleuze counterposes the "infinite representation" he sees as underpinning Leibniz's viewpoint —one "completely lacking in sense"— with the finitist interpretation of the calculus urged by Carnot, one that has now become the norm. On the finitist interpretation the "fate of the calculus" is no longer tied to infinitesimals:

> We know in effect that [1] the notion of limit has lost its phoronomic character and involves only static considerations; that [2] variability has ceased to represent a progression through all the values of an interval and come to mean only the disjunctive assumption of one value within that interval; that [3] the derivative and the integral have become ordinal rather than quantitative concepts; and finally that [4] the differential designates only a magnitude left undetermined so that it can be made smaller than a given number as required. (176; numeration added)

"The birth of structuralism at this point," he adds, "coincides with the death of any genetic or dynamic ambitions of the calculus" (176).

Now, as a statement of the received view of the status of Leibniz's infinitesimals, I think this appraisal cannot be faulted. But *I do not think it is a correct interpretation of Leibniz's own position on infinitesimals*; and, what is interesting, given Deleuze's insightful remark about the connection of modern analysis with the birth of structuralism, is that Leibniz himself upheld [4], the Archimedean interpretation of infinitesimals, while upholding the phoronomic character of the limit, the conception of physical quantities as continuously varying, and the quantitative nature of the derivative and integral, in opposition to [1]-[3]. In what follows, I shall give my interpretation of Leibniz's foundation for his calculus, showing how it is not wanting in rigour, even though the rigour is expressed from within a dynamic and not a structural rubric.[4]

[4] I will not explore the implications for an alternative to structuralism beyond this setting straight of the Leibnizian foundations, even though there is much to be said — especially about the history of the interpretation of these foundations: their appropriation by Hermann Cohen and the neo-Kantians, and also by Bergson, and how some of these authors' misconceptions paved the way for the victory of Cantor, Russell and the set-theoretic foundations that have come to dominate the philosophy of mathematics.

2. Leibniz's syncategorematic interpretation

Leibniz's attempts to explain infinitesimals as *fictions* has been seen as something of a piece of diplomacy in aid of his supporters in France such as L'Hôpital, a last ditch effort to save them from the criticisms of Rolle and Nieuwentijt. But the subtlety of Leibniz's position has been underestimated, as Hidé Ishiguro has argued in the second edition of her *Leibniz's Philosophy of Logic and Language* (1990). There she has given an insightful presentation of Leibniz's mature interpretation of infinitesimals which is completely in accord with the interpretation of his views on the infinite that I have presented above.[5] According to Ishiguro, Leibniz's position is analogous to Russell's regarding definite descriptions: "one can have a rigorous language of infinity and infinitesimal while at the same time considering these expressions as being syncategorematic (in the sense of the Scholastics), i.e. regarding the words as not designating entities but as being well defined in the proposition in which they occur" (82).

This is contrary to the usual understanding (faithfully recounted by Deleuze), where Leibniz is understood as committed to an "infinite representation". Even Henk Bos (1974/75, 54-56), whose profound contribution to the understanding of Leibniz's differential calculus is universally acknowledged, takes Leibniz to have provided two different approaches to interpreting infinitesimals. One is finitist and Archimedean, in which differentials are interpreted as finite differences that may be taken so small as to lead to an error less than any assignable. The second is based on the Law of Continuity: it accepts infinitely small quantities as "true quantities of their own sort", but insists on interpreting them as fictions. But as Ishiguro has argued, these approaches are in fact two sides of the same coin. To say that *dx* is a fiction is *not* to say that there exist "fixed entities with non-Archimedean magnitudes, the introduction of which shortens proofs" (Ishiguro 1990, 83). "The word infinitesimal does not designate a special kind of magnitude. It does not designate at all." (83) This is what is meant by calling the interpretation *syncategorematic*: terms involving infinitesimals are "ostensibly designating expressions which follow certain *sui generis* rules" (83) and whose introduction shortens proofs; but they do not in fact designate real entities. The syncategorematic interpretation explains how it is possible to treat infinitesimals *as if* they are infinitely small actuals under certain well-defined conditions. As Ishiguro puts it,

[5] Of course, Ishiguro's interpretation of Leibniz's concept of infinitesimals and the infinite as syncategorematic predates my stumbling on the same interpretation of Leibniz on the infinite (2001) by some years; she brought this to my attention with typical modesty in Berlin in 2001.

> When we make reference to infinitesimals in a proposition, we are not designating a fixed magnitude incomparably smaller than our ordinary magnitudes. Leibniz is saying that whatever small magnitude an opponent may present, one can assert the existence of a smaller magnitude. (87)

As she goes on to argue, "Leibniz denied that infinitesimals were fixed magnitudes, and claimed that [in our apparent references to them] we were asserting the existence of variable finite magnitudes that we could choose as small as we wished" (92). Leibniz's infinitesimals, that is, are in keeping with the Archimedean Axiom.

In what follows I shall argue for the correctness of Ishiguro's interpretation by reference to Leibniz's own writings on foundations. I will argue that Leibniz's characterization of infinitesimals as fictions is not a stratagem invented to save his embarrassment at the fact that the calculus of infinitesimals had worked so well despite a supposed lack of foundation. Rather, it has a precise mathematical content, perfectly consistent with his philosophy of the infinite and solution to the continuum problem (Arthur 2001a, b, 2008a, b). Moreover, I shall argue here, this content is given by the foundation of the method on the Archimedean Axiom.

3. Leibniz's Justification of the Calculus

The Archimedean Axiom[6] asserts that for any two geometric quantities x and y (with $y > x$), a natural number n can be found such that $nx > y$. This entails the corollary that no matter how small a geometric quantity is given, a smaller can be found:

> **Axiom of Archimedes**: "Magnitudes are said to have a ratio to one another when the lesser can be multiplied so as to exceed the another" (Euclid, *Elements*, Book V, Def. 4: Todhunter 1933, 134). That is, for any two geometric quantities x and y (with $y > x$), a natural number n can be found such that $nx > y$.
> **Corollary**: no matter how small a geometric quantity is given, a smaller can be found.

It is this corollary to Archimedes' axiom that precludes actual infinitesimals, where an *actual infinitesimal* may be thought of as the counterpart of the *categorematic infinite*. Just as the categorematic infinite is a number greater than all finite numbers, so an actual infinitesimal quantity may be defined as a quantity smaller than any finite quantity.

> **Actual Infinitesimal**: a quantity smaller than any finite quantity.

[6] This axiom is so called because Archimedes used it as the basis for his method of exhaustion.

This explains why actual infinitesimals are called "non-Archimedean quantities", and a geometry incorporating them a non-Archimedean system. (I have defined all these items for geometry; similar definitions can be given for numbers, so that an actual or non-Archimedean infinitesimal is a number smaller than any finite number.)

It is a mark of the huge influence of Bertrand Russell on twentieth century philosophy of mathematics that it is widely accepted that non-Archimedean infinitesimals were "driven out of mathematics once and for all, or so it seemed" by Cantor and Dedekind before they were "rehabilitated as perfectly good numbers" by the non-standard models of analysis invented by Abraham Robinson in the 1970s. As a matter of historical fact, however, despite the near total banishment of infinitesimals from the calculus, a consistent algebraic theory of non-Archimedean infinitesimal magnitudes was developed already in the 1870s, and was developed in the early decades of the twentieth century into theories of non-Archimedean ordered groups and semigroups, non-Archimedean ordered algebraic systems, and non-Archimedean geometry.[7] So it was not at all the case that mathematics had to wait for Robinson to rehabilitate infinitesimals.

Ironically, Leibniz himself entertained three more or less distinguishable theories (or theory-sketches) of actual infinitesimals prior to his invention of the calculus that bear several points of analogy with some of the modern approaches (see Arthur 2008). What compounds this irony quite deliciously, however, is the additional fact that in the very work in which he first laid out his infinitesimal methods for solving quadratures Leibniz gave a well thought-out justification for these infinitesimals on a thoroughly Archimedean foundation!

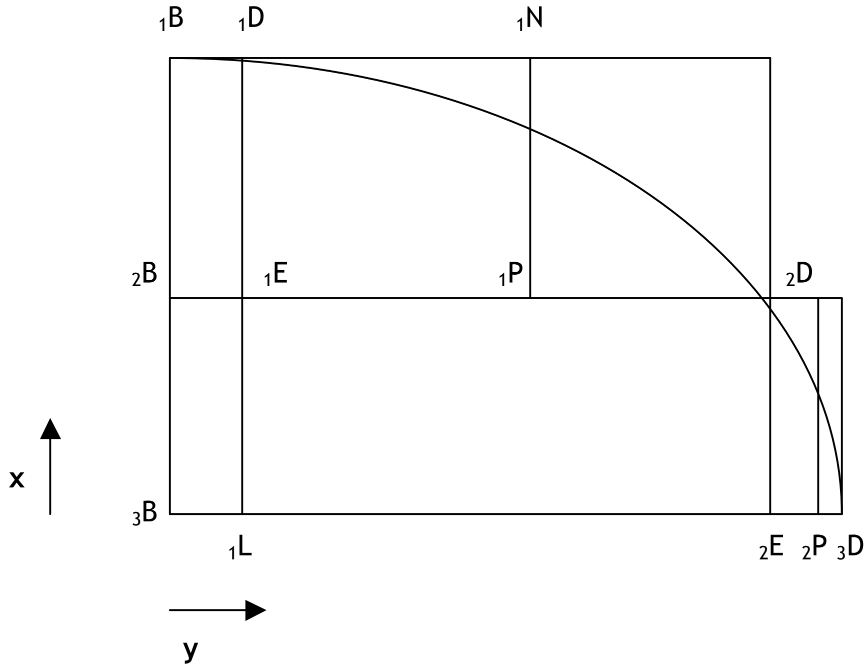
The work in question is a comprehensive Latin treatise, *De quadratura arithmetica*, written by Leibniz in 1675-76, which was only edited and published in recent times by Eberhard Knobloch (Leibniz 1993). In this treatise, as Knobloch has shown, "Leibniz laid the rigorous foundation of the theory of infinitely small and infinite quantities" (2002, 59); and Knobloch's interpretation of Leibniz's foundational work, as I have argued elsewhere (Arthur 2008a, 2009a), is fully in keeping with Hidé Ishiguro's attribution to Leibniz of a "syncategorematic" interpretation of infinitesimals. In *De quadratura* Leibniz promotes his new method of performing quadratures directly "without a *reductio ad absurdum*" (Prop 7,

[7] For an explosion of the myth that non-Archimedean systems and infinitesimals were purged from mathematics by Cantor and Weierstraß until the advent of Non-Standard Analysis see the authoritative study by Phil Ehrlich (2006).

Scholium; *De quadratura*, 1999, p. 35), by what we would now call a direct integral. This, he believes, necessarily involves the assumption of "fictitious quantities, namely the infinite and the infinitely small" (35). The traditional Archimedean method of demonstration was by a double *reductio ad absurdum*: it would be shown that a contradiction could be derived on the assumption that the quantity in question was smaller than a given value, and another contradiction on the assumption that the quantity in question was greater than that value, thus proving that it equalled it. Leibniz's method is instead to proceed by an application of the Archimedean Axiom, appealing to the corollary that no matter how small a geometric quantity is given, a smaller can be found. Thus he prefers a justification "which simply shows that the difference between two quantities is nothing, so that they are then equal (whereas it is otherwise usually proved by a double *reductio* that one is neither greater nor smaller than the other)" (35). That is, he applies the Archimedean axiom to demonstrate that the error involved in calculations with infinitely small differences can be reduced to a quantity less than any given quantity by taking a difference sufficiently small, rendering it effectively null.

Moreover, this justification does not have to be effected in every case: it is enough to show that it can be done in a suitably general case. This Leibniz does in a case that is surprisingly general, given the usual accusations about the parlous lack of justification he and Newton are alleged to have provided for their methods. For the key theorem that Leibniz successfully demonstrates in *De quadratura arithmetica* using this Archimedean method is Proposition 6, a theorem that rigorously justifies what is now known as Riemannian Integration, as Eberhard Knobloch has shown in detail (2002).[8] (Leibniz provides a similar justification for his Theorems 7 and 8). The demonstration proceeds as follows (1999, pp. 30-33).

[8] The exposition I give is indebted to Knobloch's (2002), whose simplified diagram I reproduce, while following as closely as possible Leibniz's own symbolization from the *De quadratura* (Leibniz 1999).



Leibniz first identifies and then relates the sum of the "elementary rectangles [*rectangula elementales*]" (that is the Riemannian sum of unequal rectangles by which the curve is being approximated, which we may denote Q), and that of the mixtilinear figure [*spatium gradiformis*] that is the area under the curve between two ordinates ${}_1L$ and ${}_3D$, which we may denote A.[9] Then he demonstrates that the difference between the area and the sum of the elementary rectangles, A – Q, can be no greater than the area of a certain rectangle whose height is the maximum height h_m of any of the elementary rectangles, and whose width is the distance between the two ordinates ${}_1L$ and ${}_3D$. Thus $A - Q \leq {}_1L_3D \times h_m$. But because the curve is assumed continuous, Archimedes' Axiom applies: "this greatest height (an abscissa) can be chosen smaller than any given quantity, because the curve is continuous" (65). Thus the height h_m, even though it is greater than the heights of all the other elementary rectangles, "can be assumed smaller than any assigned quantity, for however small it is assumed to be, still smaller heights could be taken." Therefore the area of the rectangle ${}_1L_3D \times h_m$ "can also be made smaller than any given surface". It therefore follows that the difference between the area under the curve and the Riemannian sum, A – Q, "can also be made smaller than any given quantity. QED." (pp. 30-33). There is therefore no error

[9] Here I am following Knobloch's symbolization, where ${}_3D$ is the greatest ordinate in place of Leibniz's ${}_4D$, and the greatest height of any of the elementary rectangles is h_m in place of his ψ_4D: in the original the sum of the elementary rectangles $\leq {}_1L_4D \times \psi_4D$ (Leibniz 1993, 29-32).

involved in calculating the quadrature as the sum of an infinity of infinitesimal areas, provided this is understood to mean that there are more little finite areas than can be assigned, and that their magnitude is smaller than any that can be assigned.

It is worth recalling at this point Deleuze's charge that "It is obvious that to invoke here the infinitely small, and the infinitely small magnitude of the error (if there is 'error'), is completely lacking in sense and prejudges infinite representation." This would only be so if the infinitesimal areas in question were non-Archimedean, or actual infinitesimals. In fact, however, what Leibniz has done is to invoke finite areas that can always be made smaller than any preset magnitude. He has justified proceeding, in this case, as if there were an infinity of infinitesimals precisely without assuming an infinite representation! It is also hard to see any difference in rigour between his justification and the finitist justifications of Carnot and Cauchy. Thus Leibniz appears justified in remarking about this theorem, "it serves to lay the foundations of the whole method of indivisibles in the soundest way possible" (1993, 24); Knobloch agrees, calling it a "model of mathematical rigour (2002, 72)".

The point here is not that Leibniz has two methods, as Bos supposed, one committed to the existence of infinitesimals and the other Archimedean;[10] nor is it the case that he simply uses the infinitesimal calculus and then airily refers to the fact that one could *instead* have used an Archimedean method. It is that, as examples like this demonstrate, the Archimedean Axiom justifies proceeding as if there are infinitesimals, and at the same time demonstrates that what they really stand for are finite quantities which can be taken as small as desired. Once this is demonstrated in a suitably general case, it also justifies the use of these fictions in other analogous cases. As Leibniz himself writes, "Nor is it necessary always to use inscribed or circumscribed figures, and to infer by *reductio ad absurdum*, and to show that the error is smaller than any assignable; although what we have said in *Props. 6, 7 & 8* establishes that it can easily be done by those means." (Scholium to *Prop. 23*, Leibniz 1993, 69)

In effect, the application of the Archimedean Axiom enables a kind of Arithmetic of the Infinite. In his article, Knobloch identifies a number of rules which are tacitly applied by Leibniz in *De quadratura*, "without demonstrating them, only relying on the 'law of continuity'" (2002, 67). Examples are

[10] Cf. Bos (1974/75) on "Leibniz's two different approaches to the foundation of the calculus" (p. 55).

1. Finite + infinite = infinite.

2. 2.1 Finite ± infinitely small = finite

   2.2 If $x = y$ + infinitely small, then $x - y \approx 0$ (is unassignable), where x and y are finite quantities. ...

It is an instructive exercise to see how these rules can be justified by the Archimedean foundation outlined above. As an example, let us take Knobloch's rule 2.2. If $x = y + dy$, where dy is an arbitrarily small variable quantity, and D is any pre-assigned difference between x and y, no matter how small, then dy may always be taken so small that $dy < D$. In particular, if D is supposed to be some fixed ultimate difference between them, then dy can be supposed smaller: so long as D and dy are quantities obeying the Archimedean axiom, the variability of dy means that it can always take a value such that $dy < D$ for any assigned D. Therefore, since the difference between x and y is smaller than any assignable, it is unassignable, and effectively null. Perfectly analogous reasoning justifies 2.1: $x + dx = x$.

Leibniz gives such an argument explicitly in a short paper dated 26 March,1676, in the same time period in which he was composing *De quadratura*:

> We need to see exactly whether it can be demonstrated in quadratures that a difference is not after all infinitely small, but nothing at all. And this will be shown if it is established that a polygon can always be inflected to such a degree that even when the difference is assumed infinitely small, the error will be smaller. Granting this, it follows not only that the error is not infinitely small, but that it is nothing at all —since, of course, none can be assumed. (A VI iii, 434; Leibniz 2001, 64-65)

Although one might wish for more perspicuous wording, Leibniz's reasoning here seems very evocative of Newton's in his justification of the method of first and ultimate ratios:

> *Quantities, or ratios of quantities, which in a given time constantly tend to equality, and before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal.* If you deny this, let their ultimate difference be D. Then they cannot approach so close to equality that their difference is less than the given difference D, contrary to hypothesis.[11]

[11] Newton (1999, 433); quoted in the version of the first edition of 1687. Ironically, when Leibniz saw Newton's *Principia* for the first time he misconstrued this *reductio*, making the rather lame remark

This consilience of the foundations of Leibniz's and Newton's justifications of quadratures is discussed more fully elsewhere (Arthur 2008a).

This is, of course, only a start to providing a satisfactory foundation for the infinitesimal methods used by Leibniz and Newton. But as I have shown elsewhere (Arthur 2009a), by a minor adjustment of Henk Bos's analysis one can use exactly the same foundation to interpret Leibniz's later (unpublished) justification of the calculus in *Cum prodiisset…* (c.1701), including Leibniz's proof of the rule for the differentiation of a product $d(xv) = xdv + vdx$. His proof does not depend on an interpretation of *dv* or *dx* as actual infinitesimals; nor are they fixed finite terms. As Bos also argues, a similar foundation can be applied for *second order differentials*, provided one is careful (as Leibniz himself was not) to stick with one "progression of the variables" (= independent variable, usually *t*).

4. Conclusion

Returning to the criticisms of Leibniz's (and Newton's) infinitesimals with which we began, it should now be clear that Earman's allegation of a contradiction is not warranted. Leibniz's differentials are finite, arbitrarily small, and variable. And given the Archimedean foundation of his methods, this is enough to ensure the rigour of proofs involving them; as well as to justify treating them as if they are infinitely small, and as if an infinity of them can sum to a finite quantity.

And as for Deleuze's criticisms, we see that Leibniz's foundation, while certainly not compatible with the modern Weierstaßian "static" or "structural" analysis, is not a theory of actual infinitesimals involving a self-contradictory "infinite representation" either, and does not relinquish its phoronomic character. Leibniz's syncategorematic interpretation provides a third approach, in a way intermediate between classical Weierstraßian analysis, which dispenses with infinitesimals, and the (consistent!) non-Archimedean theories of actual infinitesimals mentioned above. Leibniz's justification of infinitesimals as fictions is (in its own terms) a valid one.

References

that "It can be doubted whether there is an ultimate difference" (Bertoloni Meli 1993, 226), when Newton was in fact proving that there cannot be one.

Arthur R. T. W. (2001a). "Leibniz on Infinite Number, Infinite Wholes and the Whole World: A Reply to Gregory Brown", *Leibniz Review*, **11**, 103-116.

Arthur R. T. W. (2001b). "Leibniz and Cantor on the Actual Infinite," pp. 41-46 in *Nihil sine ratione*, Vol. 1, ed. Hans Poser, Berlin: Gottfried-Willhelm-Leibniz-Gesellschaft, 2001.

Arthur R. T. W. (2008a). "Leery Bedfellows: Newton and Leibniz on the Status of Infinitesimals," pp. 7-30 in Goldenbaum and Jesseph, eds., 2008.

Arthur R. T. W. (2008b). "Actual Infinitesimals in Leibniz's Early Thought", forthcoming in a special issue of *Studia Leibnitiana*, ed. Mark Kulstad and Mogens Laerke.

Arthur R. T. W. (2009a). "Leibniz's Syncategorematic Infinitesimals, Smooth Infinitesimal Analysis, and Newton's Proposition 6", forthcoming in *Infinitesimals*, ed. William Harper and Wayne C. Myrvold.

Arthur R. T. W. (2009b?). "'A complete denial of the continuous?' Leibniz's philosophy of the continuum," forthcoming in a special edition of *Synthese* devoted to the Mathematics and Philosophy of the Continuum, 2008.

Berkeley, George (1734). *The Analyst*, pp. 65-102 in *The works of George Berkeley*, IV, London/New York: T. Nelson, 1951.

Bertoloni Meli, Domenico (1993). *Equivalence and Priority: Newton versus* Leibniz. Oxford, Clarendon Press.

Brown, Gregory (2000). "Leibniz on Wholes, Unities and Infinite Number," *Leibniz Review* **10**, 21-51.

Bos, H. J. M. (1974-75). "Differentials, Higher-Order Differential and the Derivative in the Leibnizian Calculus", *Archive for the History of the Exact sciences*, **14**, 1-90.

Cantor, Georg (1932). *Gesammelte Abhandlungen*, ed. Ernst Zermelo (Berlin; repr. Hildesheim: Georg Olms, 1962).

Deleuze, Gilles (1994). *Difference and Repetition*. Columbia University Press (English transl. by Paul Patton of *Différence et Repetition* (Presses Universitaires de France, 1968).

Earman, John (1975). "Infinities, Infinitesimals, and Indivisibles: The Leibnizian Labyrinth". *Sudia Leibnitiana* **7**/2, 236-251.

Ehrlich, Philip (2006). "The Rise of non-Archimedean Mathematics and the Roots of a Misconception I: the Emergence of non-Archimedean Systems of Magnitudes," *Archive for History of Exact Sciences* 60, pp. 1-121.

Euclid, *Elements*, 134

Goldenbaum, Ursula and Douglas Jesseph, eds. (2008). *Infinitesimal Differences: Controversies between Leibniz and his Contemporaries* (Berlin: de Gruyter).

Guicciardini, Niccolò (2002). "Analysis and synthesis in Newton's mathematical work." Pp. 308-328 in *The Cambridge Companion to Newton*. Eds. I. Bernard Cohen and George E. Smith. Cambridge: Cambridge University Press.

Guicciardini, Niccolò (2003). "Conceptualism and contextualism in the recent historiography of Newton's *Principia*," *Historia Mathematica* 30, 407-431.

Ishiguro, Hidé (1990). *Leibniz's Philosophy of Logic and Language* (2nd ed.) Cambridge: Cambridge University Press.

Knobloch, Eberhard (2002). "Leibniz's Rigorous Foundation of Infinitesimal Geometry by means of Riemannian Sums." *Synthese* 133: 59-73.

Leibniz, G. W. (1675-6/1993). *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*. Ed. and commentary by Eberhard Knobloch. Göttingen: Vandenhoek & Ruprecht.

Leibniz, G. W. (1684). "*Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singulare pro illis calculi genus.*" *Acta Eruditorum*, October 467-473. GM v 220-226.

Leibniz, G. W. (1693). Letter to Foucher, January 1692; *Journal des Sçavans*, March 16, 1693, G.i.416.

Leibniz, G. W. (1701). *"Cum prodiisset atque increbuisset Analysis mea infinitesimalis …"* in Leibniz (1846), pp. 39-50.

Leibniz, G. W. (1846). *Historia et Origo calculi differentialis a G. G. Leibnitzio conscripta* (ed. C. I. Gerhardt). Hannover.

Leibniz, G. W. (2001). *The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672-1686*. Ed., sel. & transl. R. T. W. Arthur. New Haven: Yale University Press.

Leibniz, G. W. (2004). *Quadrature arithmétique du cercle, de l'ellipse et de l'hyperbole*. Ed. Marc Parmentier. Paris: J. Vrin.

Levey, Samuel (2008). "Archimedes, Infinitesimals and the Law of Continuity: On Leibniz's Fictionalism," pp. 107-133 in Goldenbaum and Jesseph, eds., 2008.

Loemker, Leroy, ed. & transl. (1969). *Leibniz: Philosophical Papers and Letters,* 2nd. ed. Dordrecht: D. Reidel.

Newton, Isaac (1999). *The Principia*: *Mathematical Principles of Natural Philosophy*. Trans. I. Bernard Cohen and Anne Whitman. Berkeley: University of California Press.

Rescher, Nicholas (1967). *The Philosophy of Leibniz*. Englewood Cliffs, NJ: Prentice Hall.

Todhunter, Isaac (1933). *The Elements of Euclid*. London: J. M. Dent & Sons.