Leery Bedfellows: Newton and Leibniz on the Status of Infinitesimals

Abstract: Newton and Leibniz had profound disagreements concerning metaphysics and the relationship of mathematics to natural philosophy, as well as deeply opposed attitudes towards analysis. Nevertheless, or so I shall argue, despite these deeply held and distracting differences in their background assumptions and metaphysical views, there was a considerable consilience in their positions on the status of infinitesimals. In this paper I compare the foundation Newton provides in his Method Of First and Ultimate Ratios (sketched at some time between 1671 and 1684, and published in the *Principia* of 1687) with that provided independently by Leibniz in his unpublished manuscript *De quadratura arithmetica* (1675-6) as well as in later writings. I argue that both appeal to a version of the Archimedean Axiom to underwrite their use of infinitesimal techniques, which must be interpreted as a shorthand for rigorously finitist methods.

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NEWTONIAN AND LEIBNIZIAN FOUNDATIONS: THE STANDARD CONTRAST

As is well known, Newton did not welcome Leibniz's efforts at establishing a differential calculus: his attitude, one might say, ranged between deep suspicion, disdain and utter hostility. In his eyes, Leibniz's differential calculus was at best a sample of the new method of analysis, an unrigorous symbolic method of discovery that could not meet the standard of rigorous proof required in geometry; and at worst, not just a plagiarism of his own work, but a dressing up and masking in Leibniz's fancy new symbols of the deep truths of his method of fluxions, which did not depend on the supposition of infinitesimals but was instead founded directly in the "real geneses of things". Leibniz, for his part, while accepting many of Newton's results, harboured doubts about Newton's understanding of orders of the infinitely small, which to his way of thinking was betrayed by the unfoundedness of Newton's composition of non-uniform with uniform motions in the limit.

There are some profound differences here in the respective thinkers' philosophies of mathematics, involving differing conceptions of proof, of the utility of symbolism, and in the conceptions of how mathematics is related to the physical world. I do not want to understate them. Nevertheless, I shall contend here, there is a very real consilience between Newton's and Leibniz's conceptions of infinitesimals, and even in the foundations they provide for the method of fluxions and for the differential calculus.

Newton's own evaluation of the difference in their methods was given by him in the supposedly "neutral" report he submitted anonymously to the Royal Society in 1715, "Account of the *Commercium Epistolicum*" (in MPN VIII). There he depicted his method as proceeding "as much as possible" by finite quantities, and as founded on these and the continually increasing quantities occurring in nature, in contrast to Leibniz's, founded on indivisibles that are inadmissible in geometry and non-existent in nature:

> We have no Ideas of infinitely little Quantities, & therefore Mr. *Newton* introduced Fluxions into his Method that it might proceed by finite Quantities as much as possible. It is more Natural & Geometrical because founded upon the *primae quantitatum nascientum rationes* w^{ch} have a Being in Geometry, whilst *Indivisibles* upon which the Differential Method is founded have no Being either in Geometry or in Nature. There are *rationes primæ quantitatum nascentium* but not *quantitates primæ nascentes*. Nature generates Quantities by continual Flux or Increase, & the ancient Geometers

> admitted such a Generation of Areas & Solids [...]. But the summing up of Indivisibles to compose an Area or Solid was never yet admitted into Geometry. (*MPN* VIII, 597-8)

This has been an influential account. Although it has long been recognized that Leibniz's differential calculus is a good deal more general than the Cavalierian geometry of indivisibles, and that Newton's characterizing of it as founded on indivisibles must be interpreted accordingly, the idea that Leibniz's methods were committed to the existence of infinitesimals has stuck. As a result, his official position that they are to be taken as fictions has been regarded as a not very successful attempt to distance himself from the foundational criticisms brought to bear by Nieuwentijt, Rolle, and the Newtonians, when in fact his method is based upon infinite sums and infinitely small differences, and thus firmly committed to infinities and infinitesimals. Newton, on the other hand, has been seen as moving from an early purely analytic method depending on a free use of infinitesimals to a mature view, represented in his Method of First and Ultimate Ratios published in the *Principia*, where (officially, at least) there are only limiting ratios of nascent or evanescent quantities, and never infinitely small quantities standing alone.

Newton's "Account" of the *Commercium Epistolicum* is a late text in his mathematical development, occurring as the culmination of a process of distancing himself from Analysis. By the 1680s he had turned away from the "moderns" in favour of Pappus and Apollonius, and an insistence on geometric demonstration. But the contrast between an early analytic Newton and the later conservative geometrician should not be overemphasized. The conception of fluxions or velocities by means of which Newton articulated what we call the "Fundamental Theorem of the Calculus" is intimately bound up with the kinematic conception of curves that he inherited from Barrow and Hobbes. Thus although Newton's first formulations of his theory of fluxions are analytic in the sense that they are couched in terms of equations and algebraic variables, his kinematic understanding of curves and surfaces already implicitly involves a notion of the quantities represented by the variables as geometric, and as generated in time.

I shall argue, accordingly, that there is not such a huge gulf between Newton's analytic method of fluxions and the synthetic methods he later developed. Moreover, I contend, when Newton comes to secure the foundations of his synthetic method in the Method Of First and Ultimate Ratios (MFUR), he appeals to Lemma 1, which is a synthetic version of the Archimedean Axiom: "Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal" (Newton 1999, 433). The axiom then serves to justify Newton's appeals to infinitesimal moments in supposedly geometric proofs such as that of Proposition 1 of the *Principia*, since these moments can be understood as finite but arbitrarily small geometric quantities in accordance with the Archimedean Axiom. Furthermore, although Newton himself is careful to apply Lemma 1 only to *ratios of quantities*, the lemma as stated by him also applies directly to *quantities*; and Leibniz will appeal to a very similar principle applied to quantities as the foundation of his own method. In fact, the principle Leibniz appeals to, which takes differences smaller than any assignable to be null, is stated independently by Newton in his analytic method of fluxions (1971), and is a straightforward application of the Archimedean Axiom.

Contrary to the standard depiction of their methods, then, there is a great similarity in the foundations of Newton's and Leibniz's approaches to the calculus. In fact, as I show by a detailed analyses of Newton's proof of Lemma 3 of his MFUR, and of Leibniz's proof of his Proposition 6 of *De quadratura arithmetica* (1676/1993), their (contemporary and independent) attempts to provide rigorous foundations for their infinitesimalist methods by an appeal to the Archimedean Axiom are in detailed correspondence, and perfectly rigorous. The rigour of Leibniz's approach to proving proposition 6 has already been stressed by Eberhard Knobloch (2002). Here I extend that analysis to show its compatibility with the *syncategorematic interpretation of infinitesimals* attributed to Leibniz by Hidé Ishiguro.

NEWTON'S MOMENTS AND FLUXIONS

The paper that is generally taken as containing Newton's first full statement of his analytic method of fluxions is "To Resolve Problems by Motion", written in October 1666 as the culmination of several redraftings (*MPN* I, 400-448). The commitment to the kinematical representation of curves is evident in its title, and this is so also for the earlier drafts out of which it develops: two drafts of "How to draw tangents to Mechanicall lines" [30? October 1665 and 8 November 1665, resp.], a third draft titled "To find y^e velocitys of bodys by y^e lines they describe", [November 13th 1665], and a fourth titled "To resolve these & such like

Problems these following propositions may bee very usefull", [May 14, 1666].[1] Thus Newton's recipe in Proposition 7 for what we, after Leibniz, call differentiation, is couched by him in terms of the velocities of bodies:

> [Prop.] 7. Haveing an Equation expressing y^{e} relation twixt two or more lines x, y, z &c: described in y^{e} same time by two or more moveing bodies A, B, C, &c [Fig. 1]: the relation of their velocitys *p*, *q*, *r*, &c may bee thus found, viz:

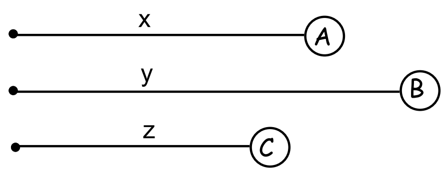


Figure 1

> Set all y^{e} termes on one side of y^{e} Equation that they may become equall to nothing. And first multiply each terme by so many times *p*/*x* as *x* hath dimensions in y^{t} terme. Secondly multiply each terme by so many times *q*/*y* as *y* hath dimensions in it. Thirdly (if there be 3 unknowne quantitys) multiply each terme by so many times *r*/*z* as *z* hath dimensions in y^{t} terme, (& if there bee still more unknowne quantitys doe like to every unknowne quantity). The summe of all these products shall be equall to nothing. w^{ch} Equation gives y^{e} relation of y^{e} velocitys *p*, *q*, *r*, &c. (*MPN* I, 402)

The first thing to notice about this algorithm is that it is not purely analytic: the equations are given a geometrical interpretation in terms of lines traced by moving bodies. Second, what results from the algorithm is not a velocity but the *ratio* of two velocities, and these velocities (say, *p* and *q*) are the instantaneous velocities of two bodies at the beginning of the moment *o* for which they are assumed to travel with that velocity.

A very simple example of applying this algorithm is provided by the result Newton quotes in his demonstration of Proposition 1 of this tract —this being perhaps the very first

[1] These drafts are given in *MPN* I on pp. 369-377, 377-382, 382-389, and 390-392, resp. The last draft was subsequently cancelled and rewritten as "To resolve Problems by motion y^{e} 6 following prop. are necessary and sufficient", dated May 16, 1666, pp. 392-399.

application of the method of fluxions in physics.[2] Proposition 1 is a statement of the Resolution of Velocities, and its demonstration depends on finding the relation between the velocities of the body *A* in two directions, towards *d* and towards *f*, as it travels along the line *gc* below, with $df \perp ac$, at the very point *a* when it reaches the perimeter of the circle. Letting $df = a$, $fg = x$, and $dg = y$, we have (since Δadf is a right triangle)

$$a^2 + x^2 - y^2 = 0$$

According to Newton's algorithm given in Proposition 7 above, we must multiply each term in *x* in the equation by $2p/x$ and each term in *y* by $2q/y$, yielding

$$2xp - 2yq = 0$$

This result is quoted by Newton in his demonstration as follows:

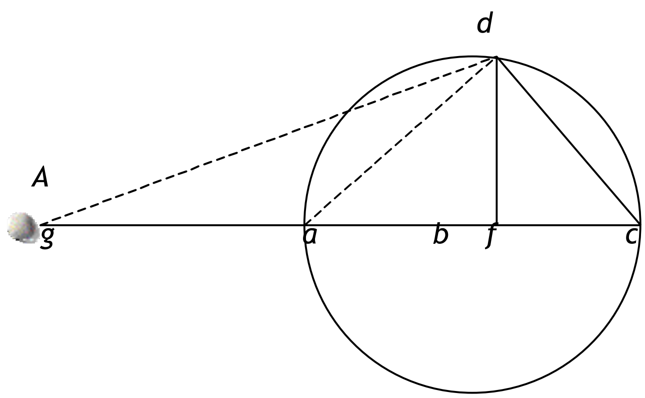


Figure 2

> Now by Prop 7th, may y^e^ proportion of (*p*) y^e^ velocity of y^t^ body towards *f*; to (*q*) its velocity towards *d* bee found *viz*: $2xp - 2yq = 0$. Or $x{:}y{::}q{:}p$. That is $gf : gd$ :: its velocity to *d* : its velocity towards *f* or *c*. & when y^e^ body *A* is at *a*, y^t^ is when y^e^ points *g* & *a* are coincident then is $ac{:}ad :: ad{:}af$ :: velocity to *c*: velocity to *d*. (415)

Or, as we would say in more Leibnizian terms, differentiating $a^2 + x^2 - y^2 = 0$ yields $2xp - 2yq = 0$, with *p* and *q* the derivatives of *x* and *y* respectively. Thus the velocities *p* and *q* are in the

2 Newton first gives the demonstration of Proposition 1 immediately after stating all 8 propositions (*MPN*, I, 415), but as Whiteside notes, Newton alludes to the fact that it can be so demonstrated in the draft of May 14th, 1666 (*MPN*, I, 390).

inverse ratio of x and y. Now when the body *A* reaches the point *a* we have $x = af$, $y = ad$, $q = v_{ad}$ and $p = v_{ac}$, yielding

$$v_{ac} : v_{ad} = ad:af$$

and since by similar triangles $ad:af = ac:ad$, we obtain finally

$$v_{ac} : v_{ad} = ac:ad \qquad \text{or} \qquad v_{ad} : v_{ac} = ad:ac$$

which, in modern notation, is the correct formula for the resolution of velocities in an oblique direction:

$$v_{ad} = v_{ac} \cos\phi, \text{ where } \phi = \angle dac.$$

Of interest to us here is the justification Newton gives in 1666 for Proposition 7. The demonstration he provides is by reference to a specific equation, $x^3 - abx + a^3 - dy^2 = 0$. There is no loss in generality in our substituting for it the above equation for Proposition 1, $a^2 + x^2 - y^2 = 0$. Newton first supposes two bodies *A* and *B* moving uniformly, the one from *a* to *c*, *d*, *e*, *f*, the other from *b* to *g*, *h*, *k*, *l*, in the same time. Then the pairs of lines *ac* and *bg*, *cd* and *gh*, *de* and *hk* etc. are "as their velocitys" *p* and *q*. (414):

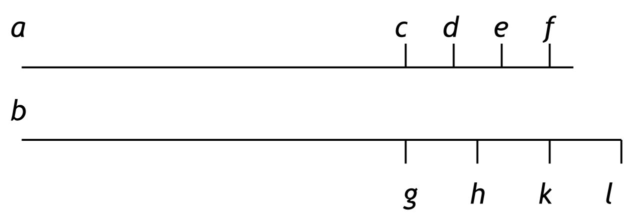


Figure 3

He then reasons that :

> And though they move not uniformely, yet are y^{e} infinitely little lines w^{ch} each moment they describe, as their velocitys w^{ch} they have while they describe y^{m}. As if y^{e} body *A* wth y^{e} velocity *p* describe y^{e} infinitely little line (*cd* =) $p \times o$ in one moment, in y^{t} moment y^{e} body *B* wth y^{e} velocity *q* will describe y^{e} line (*gh* =) $q \times o$. For $p:q :: po:qo$. Soe y^{t} if y^{e} described lines bee (*ac* =) *x*, & (*bg* =) *y*, in one moment, they will bee (*ad* =) $x + po$, & (*bh* =) $y + qo$ in y^{e} next. (414)

Now he claims that "I may substitute $x + po$ & $y + qo$ into y^e^ place of x & y; because (by y^e^ lemma) they as well as x & y, doe signify y^e^ lines described by y^e^ bodys A & B" (414). Thus for the equation $a^2 + x^2 - y^2 = 0$ we get

$$a^2 + x^2 + 2pox + p^2o^2 - y^2 - 2qoy - q^2o^2 = 0$$

Subtracting the original equation gives

$$2pox + p^2o^2 - 2qoy - q^2o^2 = 0$$

"Or dividing it by o tis [$2px + p^2o - 2qy - q^2o = 0$]. Also those termes are infinitely little in w^ch^ o is. Therefore omitting them there rests

$$[2xp - 2yq = 0]$$

The like may bee done in all other equations" (*MPN* I, 414-15)

Here Newton's division by o prior to omitting terms in o because they are "infinitely little" is, of course, lacking in rigour. Either, one may object, adding po to x takes body A to "y^e^ next" point on the line representing its path, and one is committed to composing that line out of successive infinitesimal linelets (and thus succumbing to the paradoxes of the continuum); or indeed, $x + po$ does not at all differ from x, in which case division by o is completely illegitimate. And yet Newton's algorithm is framed in terms of *ratios* of quantities and their velocities in the moment o. Of course, there is no way to represent an instantaneous velocity geometrically save by showing the line segment (*cd* in figure 3) that a body would cover if it continued with that velocity for a time o. From this point of view, the moment o is more nearly a device enabling instantaneous velocities to be geometrically represented: po is the distance the body A would have covered if it had proceeded with the velocity p for some time o. The ratio po:qo is of course equal to the ratio of p and q for any finite o. Moreover, it is implicit in the kinematic representation that the velocities p and q are the velocities at the very beginning of the moment o, so that the term for po:qo calculated by Newton's algorithm, which will still generally contain terms in o, will be closer to p:q the closer o is to 0. The justification for neglecting the remaining terms in o is therefore not so much that they are conceived as "infinitely little" in the sense of absolutely infinitely small, but in the sense that the ratio p:q = po:qo represents the ratio of p and q

right at the beginning of the moment, so that the smaller *o* is made, the smaller will be the terms still containing *o*, and the more nearly will the resulting expression represent the ratio.

Thus in the context of a kinematic and geometric interpretation of the quantities involved, Newton's early appeal to the infinitely small cannot simply be taken as committing him to a composition of quantities out of infinitesimals. In fact, his procedure already implicitly involves a kind of limiting process: to find the ratio of the velocities precisely at the beginning of the moment *o* (e.g. at the instant the moving body *A* reaches the point *a* in the above diagram), *o* must be shrunk to zero, so that the extra terms in the expression of this ratio still depending on the quantity *o* will therefore also vanish, with the resulting expression yielding the "first ratio" of these velocities.

Newton himself recognized this soon enough, and proceeded to make the limit concept implicit in the kinematical representation the foundation of the synthetic method of fluxions. He drew up these early results, as well as those outlined in his *De Analysi per æquationes numero terminorum infinitas* (1669; publ. 1711; in *MPN* I,), into a formal Latin treatise intended for publication, the *Tractatus de methodis serierum et fluxionum* (1671; publ. 1736; *MPN* I,), or "Treatise on Fluxions" for short, where the terminology of "fluxions" was first introduced. But he remained unsatisfied with the foundations of his methods, and in an Addendum on "The Theory of Geometrical Fluxions" made just after completing the latter, he developed a wholly synthetic approach, "based on the genesis of surfaces by their motion and flow". (*MPN*, III, 328-31; Guicciardini 2002, 315.) Axiom 4 of this Addendum was "Contemporaneous moments are as their fluxions" (*MPN*, III, 330), or more perspicuously perhaps, "Fluxions are as the contemporaneous moments generated by those fluxions" (draft). Whiteside comments: "This fundamental observation opens the way to subsuming limit-increment arguments as fluxional ones, and conversely so" (MPN III, 330, fn 7.)

As Guicciardini has noted (2002, 414-17), these foundations are synthetic in two distinct senses: they are based on explicit axioms from which propositions are derived, "synthesis" as opposed to "analysis"; and the quantities are not the symbols but fluent geometrical figures, synthetic in the sense of flowing, increasing, staying constant, or decreasing continuously in time. The emphasis on synthesis (in this dual sense) is a symptom of Newton's progressive disenchantment with analysis in the 1670s, and a growing respect for the geometry of the ancients. This process is taken further in *Geometria curvilinea*, written some time between 1671 and 1684, where Newton stresses the generation of geometric quantities in time:

> Those who have measured out curvilinear figures have usually viewed them as consisting of infinitely many infinitely small parts. But I will consider them as generated by growing, arguing that they are greater, equal or smaller according as they grow more swiftly, equally swiftly or more slowly from the beginning. And this swiftness of growth I shall call the fluxion of a quantity. [*MPN* IV, 422-23]

This interpretation of his mathematics explains the contrast Newton draws between the ontological foundation of his methods ("This Method is derived immediately from Nature her self") and the lack of such a foundation in the analysis of Leibniz. It is emphasized even more strongly in the *De quadratura curvarum* of 1693, where Newton writes:

> I don't here consider Mathematical Quantities as consisting of *indivisibles*, whether least possible parts or infinitely small ones, but as described by a continual motion. Lines are described, and by describing are generated, not by any apposition of Parts, but by the continuous motion of Points, Surfaces by the motions of Lines, Solids by the motion of Surfaces, Angles by the Rotation of their Legs, Time by a continual Flux, and so on in all the rest. These *Geneses* are founded upon Nature, and are every Day seen in the motions of Bodies. (Newton 1964, 141)

In these passages Newton not only claims that geometric quantities are founded *in rerum natura*, he also explicitly repudiates their composition out of infinitely small parts (infinitely small quantities have "no Being either in Geometry or in Nature"). As he had come to recognize, the moments of quantities do not have to be supposed as infinitely small quantities, falling outside the scope of geometry based on the Archimedean Axiom, but can instead stand for finite quantities that can be taken as small as desired. This is the foundation of his synthetic method of fluxions Newton presents in the *Geometria curvilinea*, and which he will publish in the *Principia* under the new moniker the Method of First and Ultimate Ratios. Although infinitely small quantities still occur in Newton's mature work, they are interpreted as standing for finite but small quantities that are on the point of vanishing, with the ratio between two such quantities remaining finite in this temporal limit.

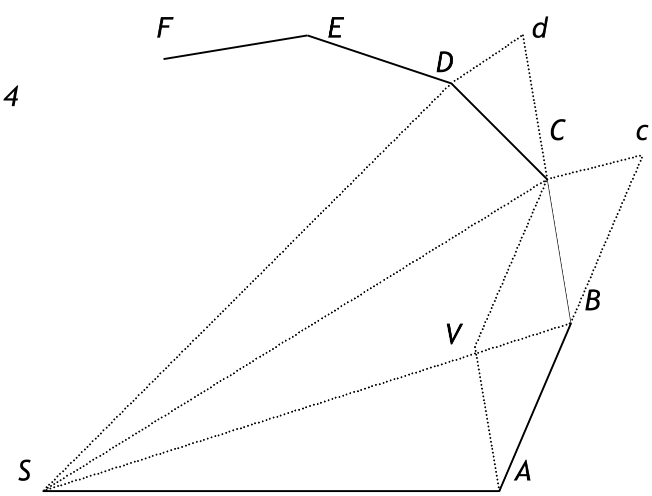
An example of this synthetic method of fluxions, I claim, is provided by Newton's demonstration of Proposition 1, in Book 1 of the *Principia*. In fact, this proposition provides a particularly good specimen of the advantages of the synthetic method of fluxions. For not only is the proof extremely economical compared to any analytic derivation of Kepler's Area

Law, it also depends on no assumptions about the nature of the force except that it is continuous and centrally directed.[3] Newton's demonstration goes as follows:

> Let the time be divided into equal parts, and in the first part of the time let a body by its inherent force describe the straight line *AB*. In the second part of the time, if nothing hindered it, this body would (by law 1) go straight on to *c*, describing line *Bc* equal to *AB*, so that —when radii *AS*, *BS* and *cS* are drawn to the centre— the equal areas *ASB* and *BSc* would be described. But when the body comes to *B*, let a centripetal force act with a single but great impulse and make the body deviate from the straight line *Bc* and proceed in the straight line *BC*. (Newton 1999, 444)

Newton now completes the parallelogram *VBcC* to compute where the body would end up under the joint action of the inertial force and the force impressed at *B* by applying the parallelogram law (corollary 1), and uses elementary geometry to prove the equality of the triangles SAB and SBC. The motion along BC will now be the new inertial motion, and the same reasoning can be applied to triangles SBC and SCD, etc.

Figure 4



> Now let the number of triangles be increased and their width decreased indefinitely, and their ultimate perimeter *ADF* will (by lem. 3, corol. 4) be a curved line; and thus the centripetal force by which the body is continually drawn back from the tangent of this curve will act uninterruptedly, while any areas described, *SADS* and *SAFS*, which

[3] Also, of course, as explained by Nauenberg (2003, 450), the curvature of the curve must remain finite, and the radius vector cannot become tangential to it.

are always proportional to the times of description, will be proportional to the times in this case. *Q.E.D.* (444)

Crucial in this proof is the appeal to Lemma 3, Corollary 4: "And therefore these ultimate figures (with respect to their perimeters *acE*) are not rectilinear, but curvilinear limits of rectilinear figures" (434).[4]

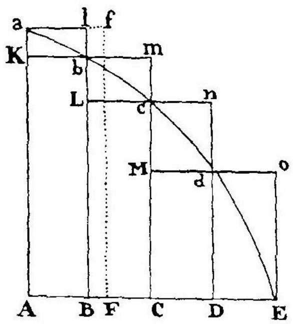


Figure 5

Lemma 3 itself is: "*the ultimate ratios [which the inscribed figure* AK*b*L*c*M*d*D, *the circumscribed figure* A*albmcndoE*, *and the curvilinear figure* A*abcd*E *have to one another] are also ratios of equality when the widths* AB, BC, CD, ... *of the parallelograms are unequal and are all diminished indefinitely.*" (433) Newton uses this result to argue in Corollary 1 that "the ultimate sum of the vanishing parallelograms coincides with the curvilinear figure in its every part", in Corollaries 2 and 3 that the figure comprehended by the chords or the tangents of the vanishing arcs "coincides ultimately with the curvilinear figure", and in Corollary 4 that "therefore these figures (with respect to their perimeters *ac*E) are not rectilinear, but curvilinear limits of rectilinear figures." (1999, 434). Thus by a similar argument the triangles in Figure 4 are not rectilinear, but curvilinear limits of rectilinear figures, the ratios between any two of which are equal.

Let us now turn to Newton's justification of this Lemma. He demonstrates it by reference to the same figure used for all the first 4 Lemmas. Having proved Lemma 2 on the supposition of equal intervals AB, BC, DE, etc., he now supposes them unequal, and lets "AF

[4] Michael Nauenberg was the first to draw attention to the importance of this lemma in Newton's justification of Proposition 1. See Nauenberg (2003, pp. 444ff.) and Arthur (2006). A minor oddity of this appeal to Lemma 3 is that the figure for Lemma 3 involves curvilinear limits of rectangles under the curve, rather than curvilinear limits of the triangles subtended under it in Proposition 1. But this does not undermine the appeal to this Lemma, since in principle the same arguments can be run for triangular rather than rectangular elements.

be equal to the greatest width" of any of the rectangles. Hence FA*af* is at least as wide as any of the rectangles, and its total height will be the sum of the heights of the differences between the circumscribed and inscribed figures. "This parallelogram will therefore be greater than the difference of the inscribed and circumscribed figures; but if its width AF is diminished indefinitely, it will become less than any given rectangle. Q.E.D." (434)

The last step of this proof is an application of Lemma 1 of the Method of First and Ultimate Ratios, which I quote here in its original wording from the First Edition:

> *Quantities, and also ratios of quantities, which in a given time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal.* (434)

Here one might object that an infinitesimal is precisely a quantity that is "less than any given quantity", so that if there exist non-zero infinitesimals then such a difference will be non-zero. In default of some further premise, the argument therefore seems to assume what it needs to prove. The missing premise is that in order for the quantities to count as geometrical quantities, they must obey the Archimedean Axiom:

> If a and b are two line segments (or other continuous geometric quantities) with $a < b$, we can always find a (finite) number n such that $na > b$.

This axiom rules out the existence of an actual infinitesimal quantity, such as the "difference less than any given quantity" mentioned in Lemma 1. As Newton argues in his demonstration of the Lemma:

> If you deny this, let their ultimate difference be D. Then they cannot approach so close to equality that their difference is less than the given difference D, contrary to the hypothesis. (433)

The "hypothesis" in question here is that they can always "approach so close to one another that their difference is less than any given quantity". This is simply an expression in synthetic form of the Archimedean Axiom: given two quantities whose difference D is less than some quantity a, we can always find a number n such that $n\mathrm{D} > a$, so that $c = a/n < \mathrm{D}$.

In fact, if we explore the origins of Lemma 1 of the MFUR we can trace a direct line of descent from the "Treatise on Fluxions". Two paragraphs of this are rewritten into the

Addendum on Geometrical Fluxions, the latter is reworked into the *Geometria curvilinea*, and it is from this that the Method of First and Ultimate Ratios is derived. The first of the two paragraphs of the "Treatise on Fluxions" runs:

> This method for proving that curves are equal or have a given ratio by the equality or given ratio of their moments, I have used because it has an affinity with methods usually employed in these cases; but a method based on the genesis of surfaces from the motion of their flowing seems more natural... (MPN III, 282)

This is transcribed to the Addendum, with the addition "..., one which will prove to be more perspicuous and elegant if certain foundations are laid out in the style of the synthetic method; such as the following:" (MPN III, 328-330), and this introduces the axioms and theorems that constitute the synthetic method. But the previous method referred to in this paragraph, that of proving "through the equality of moments", is described in the immediately preceding paragraph of the Treatise as follows:

> In demonstrations of this sort it should be observed that I take those quantities to be equal whose ratio is one of equality. And a ratio of equality is to be regarded as one which differs less from equality than any unequal ratio that can be assigned. Thus in the preceding demonstration I set the rectangle $Ep \times Ac$, that is, $Feqf$, equal to the space $FEef$ since (because their difference Eqe is infinitely smaller than them, i.e. nothing with respect to them), they have no ratio of inequality. And for the same reason, I set $Dp \times HI = HIih$, and likewise in the others. (MPN III, 282)

The principle appealed to here is this:

> If an inequality is such that its difference from a strict equality can be made smaller than any that can be assigned, it can be taken for an equality.

Let us call this the *Principle of Unassignable Difference*. This principle, clearly, is the analytic equivalent of the chief synthetic axiom, Lemma 1 of the Method of First and Ultimate Ratios. And like that Lemma, it derives its warrant from the Archimedean Axiom. This common warrant underwrites the equivalence between the analytic and synthetic methods of fluxions, allowing the translatability of statements given in terms of "indivisibles" (i.e. infinitesimals) into fluxional terminology, thus justifying Newton's claim in the *Principia* that having reduced the propositions there to the limits of the sums and ratios of First and Ultimate ratios of nascent and evanescent quantities, he had thereby "performed the same thing as by the method of indivisibles". He continues:

> Accordingly, whenever in what follows ... I use little curved lines in place of straight ones, I wish it always to be understood that I mean not indivisibles but evanescent divisible quantities, and not the sums and ratios of determinate parts, but the limits of such sums and ratios; and that the force of such demonstrations always rests on the method of the preceding lemmas. (Newton 1999, 441-2; trans. slightly modified)

Leibniz's Syncategorematic Infinitesimals

Now let us turn to Leibniz. During the same period (1671-1684) in which Newton was perfecting his synthetic interpretation of the results he had obtained in 1666, Leibniz was independently developing the algorithms and techniques he was to present as the differential and integral calculus. In his approach to the development and application of his calculus, Leibniz often stressed the pragmatic utility of his techniques, and how they could be exploited by mathematicians without their having to trouble themselves with foundational problems. These comments, together with the lack of clarity regarding foundations in his early publications, and his late pronouncements on the nature of infinitesimals precipitated by the controversy involving Rolle, Nieuwentijt and Varignon, have conspired to produce the impression that Leibniz developed his calculus without much attention to its foundations.

But this impression is entirely mistaken. For just as Newton had attempted to strengthen the foundations of his methods in his Latin treatise *De methodis serierum et fluxorum* in 1671, and again in *Geometria curvilinea* not long afterwards, so in 1675-76 Leibniz had also written a comprehensive Latin treatise on his infinitesimal methods, *De quadratura arithmetica*, which has only recently been edited and published by Eberhard Knobloch (Leibniz 1993); and in this treatise, as Knobloch has shown, "Leibniz laid the rigorous foundation of the theory of infinitely small and infinite quantities" (2002, 59). I have argued elsewhere (Arthur 2006a) that Knobloch's interpretation of Leibniz's foundational work is fully in keeping with Hidé Ishiguro's attribution to Leibniz of an interpretation of infinitesimals as "syncategorematic". That is, as I have tried to show, Leibniz's mature interpretation of infinitesimals as "fictions" has a precise mathematical content, perfectly consistent with his philosophy of the infinite and solution to the continuum problem (Arthur 2001, 2006b). Moreover, I shall argue here, this content is given by the foundation of the method on the Archimedean Axiom. Thus Leibniz's justification of his infinitesimal methods will be seen to be in surprising conformity with Newton's.

As regards foundations, the nub of the *De quadratura arithmetica* occurs in Proposition 6 (pp. 28-36), as Eberhard Knobloch has explained. Leibniz himself describes it as "most thorny; in it, it is demonstrated in fastidious detail that the construction of certain rectilinear and polygonal step spaces can be pursued to such a degree that they differ from one another or from curves by a quantity smaller than any given, which is something that is most often [simply] assumed by other authors. Even though one can skip over it at first reading, it serves to lay the foundations for the whole method of indivisibles in the sou8ndest possible way" (p. 24). The "thorniness" is evident from *Figure 6* below, (fig. 3 in the *Dqa*):

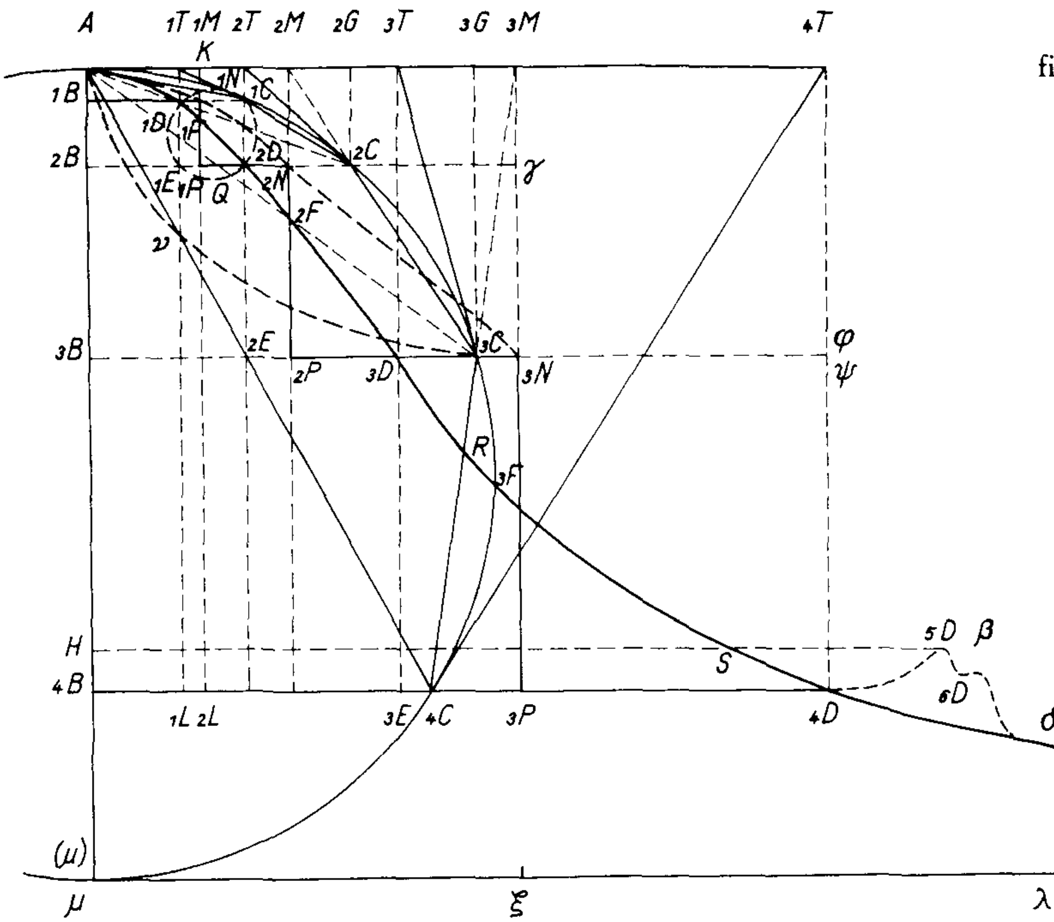


fig. 3

In this figure, the x-axis is vertical, and the y-axis is the horizontal axis across the top. The curve considered here is a circular arc C, the tangents to which at successive points on this curve (${}_1C$, ${}_2C$, ${}_3C$, ${}_4C$) cut the y-axis at the points ${}_1T$, ${}_2T$, ${}_3T$, ${}_4T$. Now a second, auxiliary curve D is defined by the points of intersection of these tangents to C with the ordinates ${}_1B$, ${}_2B$, ${}_3B$, ${}_4B$, yielding the points ${}_1D$, ${}_2D$, ${}_3D$, ${}_4D$, on this new curve. The secants joining successive pairs of points on the original curve, ${}_1C_2C$, etc., are extended to cut the y-axis in the points ${}_1M$, ${}_2M$, ${}_3M$, ${}_4M$. The points of intersection of the perpendiculars from these points M down through the ordinates B of the original curve define another set of points ${}_1N$, ${}_2N$, ${}_3N$, ${}_4N$. Provided certain conditions are satisfied —continuity, no point of inflection, no point with a vertical tangent— this construction is always possible, and as Knobloch comments, "once the second curve has been constructed, the first curve can be omitted."[5]

Following Knobloch, we will now give a simplified figure depicting a portion of the area under the curve D between the ordinates ${}_1B$ and ${}_3B$:

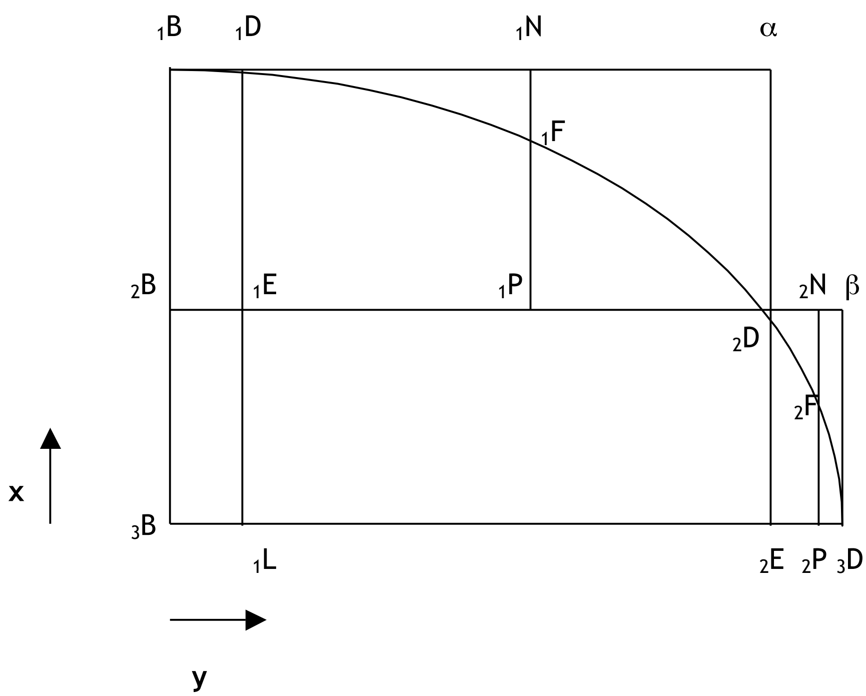


Figure 7

The demonstration of Proposition 6 then proceeds in eight numbered stages. First Leibniz partitioned the interval containing the area under the curve D is into a finite number of unequal subintervals (in the above figure there are two, ${}_1B_2B$ and ${}_2B_3B$). The rectangles

[5] See Knobloch (2002, 63), for a discussion of these conditions.

bounded by the ordinates, the x-axis to the left, and the normals through N to the right, here ${}_1B_1N_1P_2B$ and ${}_2B_2N_2P_3B$, he called *elementary rectangles*; the rectangles overlapping these bounded by successive points on the curve, here ${}_1D\alpha_2D_1E$ and ${}_2D\beta_3D_2E$, he called *complementary rectangles*.

In stage 2, he computed the (absolute value of the) difference between the area under the mixtilinear figures ${}_1B_1D_2D_2B$ and ${}_2B_2D_3D_3B$, and their corresponding elementary rectangles ${}_1B_1N_1P_2B$ and ${}_2B_2N_2P_3B$. In each case this difference is less than the corresponding complementary rectangle: $|{}_1B_1D_2D_2B - {}_1B_1N_1P_2B| < {}_1D\alpha_2D_1E$, etc. This is proved in stage 3 by subtracting from each their common part, ${}_1B_1D_1F_1P_2B$, etc, leaving a difference of two trilinear areas. Even the sum of these two areas is less than the complementary rectangle, so their difference certainly is. Thus $|{}_1B_1D_2D_2B - {}_1B_1N_1P_2B| = |{}_1D_1N_1F - {}_1F_2D_1P| < {}_1D\alpha_2D_1E$, etc. In step 4, it is shown that this inequality holds for all such differences between curvilinear areas and their corresponding elementary rectangles. As Knobloch has shown, Leibniz is here implicitly appealing to the triangle inequality $|A| - |B| \leq |A - B|$ (2002, 65).

Therefore (stages 5 and 6) the absolute value of the difference between the sum M of all the mixtilinear areas (the area under the curve, called by Leibniz the "total Quadrilineal") and the sum E of all the elementary rectangles approximating the area under the curve (the Riemannian sum, called by Leibniz the "step space [*spatium gradiforme*]") is less than the sum C of all the complementary rectangles: $M - E = C$. But the sum C of all the complementary rectangles ${}_1D\alpha_2D_1E$, ${}_2D\beta_3D_2E$, etc. would be less than the sum of all their bases times their common height, if all the ordinates were equally spaced. Since by hypothesis they are not, let the greatest height (say, the difference between successive ordinates ${}_3B$ and ${}_4B$) be h_m. The sum of all the bases is the difference between the greatest and smallest ordinate, ${}_1L_3D$. Therefore C is smaller than the rectangle ${}_3B\,{}_4B_1L_3D$, i.e. $C < {}_1L_3D \bullet h_m$. Hence, since $M - E = C$, we have

$$M - E < {}_1L_3D \bullet h_m,$$

where h_m is the greatest height of any of the elementary rectangles.

But (stage 7) the abscissa representing this greatest height, "even though it is greater than, or at any rate not less than, any of the other intervals assumed, can nevertheless be assumed smaller than any assigned quantity; for however small it is assumed to be, others can be assumed still smaller." (31-32). Therefore "it will follow that *the rectangle [$_3B\,_4B_1L_3D$]*, having a height which can be assumed smaller than any given line, also *can be made smaller than any given surface.*"

It therefore follows (stage 8) that "*the difference between this Quadrilineal* (which is the subject of this proposition) *and the step space* [i.e. *M* – *E*] *can be made smaller than any given quantity.*" (32) That is, the difference between the Riemannian sum and the area under the curve is smaller than any assignable, and therefore null.

As Leibniz points out, the prolixity of this proof is due in part to the fact that it is considerably more general than the "common method of indivisibles", where one is "compelled for safety's sake, as was Cavalieri, to restrict the method to parallel ordinates, and to suppose that the intervals between any two successive ordinates are always equal" (69). In that case the points N and the points D coincide and "the demonstration is far easier" (32), as he proceeds to show.

Several things about this demonstration are worthy of note. As Leibniz observes in the Scholium to Proposition 7, "the demonstration has the singular feature that the result is achieved not by inscribed and circumscribed figures taken together, but by inscribed ones alone" —although more accurately, the step figure is, as Knobloch says, "something in between" an inscribed and a circumscribed one (2002, 63). Leibniz's method, in fact, is extremely general and rigorous; the same construction of elementary and complementary rectangles could be constructed for any curve whatsoever satisfying the three conditions outlined. It amounts in modern terms to a demonstration of "the integrability of a huge class of functions by means of Riemannian sums which depend on intermediate values of the partial integration intervals" (Knobloch, 2002, 63).

Second, it is strictly finitist. As Leibniz observes, the traditional Archimedean method of demonstration was by a double *reductio ad absurdum*. But his preference is instead to proceed by a direct *reductio* to prove that "the difference between two quantities is nothing". As he explains in the continuation of the Scholium to Prop. 7,

> For my part I confess that there is no way that I know of up till now by which even a single quadrature can be perfectly demonstrated without an inference *ad absurdum*. Indeed, I have reasons for doubting that this would be possible through natural means without assuming fictitious quantities, namely, infinite and infinitely small ones; but

> of all inferences *ad absurdum* I believe none to be simpler and more natural, and more proper for a direct demonstration, than that which not only simply shows that the difference between two quantities is nothing, so that they are then equal (whereas otherwise it is usually proved by a double reductio that one is neither greater nor smaller than the other), but which also uses only one middle term, namely either inscribed or circumscribed, rather than both together. (Leibniz 1993, 35)

We see here a distinction between the method of integration using infinitely many infinitely small elements, which Leibniz characterizes as fictitious, and the direct *reductio ad absurdum* method just exploited in the demonstration above. As we saw there, this involves an inference from the fact that a difference between two quantities can be made smaller than any that can be assigned, to their difference being null. This is a *reductio* in the sense that whatever minimum difference one supposes there to be, one can prove that the difference is smaller. As we have seen, that is the very same reasoning Newton appeals to in his *Principia* to demonstrate Lemma 1 of his Method of First and Ultimate Ratios.

Third, Leibniz's demonstration of Proposition 6, just like Newton's Lemmas 1-4, licenses his infinitesimal techniques in quadratures, "laying the foundations of the whole method of indivisibles in the soundest possible way" (24). The term "indivisible" here needs to be taken with a pinch of salt: Leibniz is clear that "there is a profound difference between the indivisible and the infinitely small", and that "The Geometry of Indivisibles is fallacious unless it is explicated by means of the infinitely small; for truly indivisible points may not safely be applied, and instead it is necessary to use lines which, although infinitely small, are nevertheless lines, and therefore divisible." (Scholium to Proposition 11, 133) [6]. In Proposition 7, explaining that by "the sum of the straight lines applied to a certain axis" he means "the area of the figure formed by this continued application", he comments:

> For whatever properties of such a sum could be demonstrated by taking the interval arbitrarily small, will also be demonstrated of the curvilinear area ${}_0C\,{}_0B_3B_3C_0C$, since, if the interval is taken sufficiently small, this sum could be such that its difference from the sum of the rectangles will be smaller than any given. And so anyone contradicting our assertion could easily be convinced by showing that the error is smaller than any assignable, and therefore null. (Leibniz 1993, 39)

[6] This Scholium to Proposition 11 is recorded as deleted by Knobloch (Leibniz 1993, 132-33), but is included in the main text without comment in the edition of Parmentier (Leibniz 2004, 96-101).

This is precisely the same as the principle appealed to by Newton to found his analytic method of fluxions, which I called above the Principle of Unassignable Difference; it is simply an application of the Archimedean Axiom.

Fourth, Leibniz is explicit that the equivalence between a proof effected by infinitesimals and the corresponding rigorous kind of proof from first principles given in Proposition 6, means that infinitesimals can always be taken as a kind of shorthand for the arbitrarily small finite lines occurring in the latter. Acknowledging his free use of infinite and infinitely small quantities in proving his results concerning the circle, the ellipse and the infinite hyperboloid, Leibniz writes in the Scholium to Proposition 23 (Leibniz 1993, 69):

> The things we have said up to now about infinite and infinitely small quantities will appear obscure to some, as does anything new; nevertheless, with a little reflection they will be easily comprehended by everyone, and whoever comprehends them will recognize their fruitfulness. Nor does it matter whether there are such quantities in the nature of things, for it suffices that they be introduced by a fiction, since they allow economies of speech and thought in discovery as well as in demonstration. Nor is it necessary always to use inscribed or circumscribed figures, and to infer by *reductio ad absurdum*, and to show that the error is smaller than any assignable; although what we have said in *Props. 6, 7 & 8* establishes that it can easily be done by those means. Moreover, if indeed it is possible to produce direct demonstrations of these things, I do not hesitate to assert that they cannot be given except by admitting these fictitious quantities, infinitely small or infinitely large (see above, Scholium to *Prop 7.*)

An infinitesimal, therefore, is simply a shorthand for a quantity that may be taken as small as desired; likewise an infinite quantity is a quantity "greater than any assignable by us, or greater than any number that can be designated". Both are, with respect to Geometry, *fictions*. On whether they can be found in Nature, Leibniz is here agnostic; but "for Geometry it suffices to demonstrate what follows from their supposition." (Scholium to Proposition 11; Leibniz 1993, 133; 2004, 98)

This interpretation, as I have argued elsewhere (2001, 2006a, b), is completely in accord with the insightful presentation of Leibniz's mature interpretation of infinitesimals given by Hidé Ishiguro in the second edition of her *Leibniz's Philosophy of Logic and Language* (1990). According to Ishiguro, Leibniz held, analogously to Russell's position regarding definite descriptions, "that one can have a rigorous language of infinity and infinitesimal while at the same time considering these expressions as being syncategorematic (in the sense of the Scholastics), i.e. regarding the words as not designating entities but as being well defined in the proposition in which they occur" (82). As she goes on to argue, "Leibniz denied that infinitesimals were fixed magnitudes, and claimed that [in our apparent references to them] we were asserting the existence of variable finite magnitudes that we could choose as small as we wished" (92). This is indeed the case, as we have seen.

There is, of course, much more to say on Leibniz's syncategorematic interpretation, in particular, concerning the philosophical status of infinitesimals as fictions. Other contributors to this volume, including Ishiguro herself, will have more to say here on such issues. But I think it will be instructive for me to close by showing how Leibniz's use of infinities and infinitesimals can be justified by the Archimedean foundation he shared with Newton. Eberhard Knobloch has identified twelve rules occurring in his treatise that may be said to constitute Leibniz's "arithmetic of the infinite" (Knobloch 2002, 67-8). In the interests of space I shall just consider a small sample. The first of these rules is "Finite + infinite = infinite." Rule 2.1 is "Finite ± infinitely small = finite", and Rule 2.2 is "If $x = y$ + infinitely small, then $x - y \approx 0$ (is unassignable)" where x and y are finite quantities.

Let us take 2.2 first. If $x = y + dy$, where dy is an arbitrarily small finite variable quantity, and D is any pre-assigned difference between x and y, no matter how small, then dy may always be taken so small that $dy < D$. In particular, if D is supposed to be some fixed ultimate difference between them, then dy can be supposed smaller: so long as D and dy are quantities obeying the Archimedean axiom, the variablility of dy means that it can always take a value such that $dy < D$ for any assigned D. Therefore, since the difference between x and y is smaller than any assignable, it is unassignable, and effectively null. The same reasoning justifies 2.1.

Leibniz gives such an argument explicitly in a short paper dated 26 March, 1676:

> We need to see exactly whether it can be demonstrated in quadratures that a difference is not after all infinitely small, but nothing at all. And this will be shown if it is established that a polygon can always be inflected to such a degree that even when the difference is assumed infinitely small, the error will be smaller. Granting

> this, it follows not only that the error is not infinitely small, but that it is nothing at all —since, of course, none can be assumed. (A VI iii, 434; Leibniz 2001, 64-65)

Notable here is his claim that this argument works even if the difference D is assumed infinitely small; it does so, of course, only if the variable *dy* obeys the Archimedean axiom.[7]

To prove Knobloch's Rule 1, suppose *dz* is another arbitrarily small finite variable quantity such that the ratio *dy*:*dz* remains finite as *dz* is made arbitrarily small. Now again suppose $x = y + dy$, and divide all through by *dz*, and let *dz* become arbitrarily small. As it does so, x/dz and y/dz will each become arbitrarily large; indeed, no matter how large a quantity Q is given, *dz* can be taken sufficiently small that x/dz and y/dz will each exceed it. Thus x/dz and y/dz will each be greater than any given quantity Q, and thus infinite by Leibniz's definition, while *dy*:*dz* remains finite, yielding rule 1, Finite + infinite = infinite. Similar justifications can be given for Knobloch's other rules.

This is, of course, only a start to providing a satisfactory foundation for the infinitesimal methods used by Leibniz and Newton. In particular, it does not treat all those issues surrounding second- and higher-order infinitesimals; although, as I have argued elsewhere (Arthur 2006a), it is possible to give a successful account of second-order infinitesimals on Leibniz's syncategorematic interpretation. But this must suffice for present purposes.

Comparison: A Consilience of Foundations

In the foregoing discussion we have seen a consilience in the foundational writings of Newton and Leibniz that is quite remarkable. Not only does each thinker appeal to the Archimedean Axiom in the form of the Principle of Unassignable Difference (or its synthetic counterpart, Lemma 1) to justify methods that apparently appeal to infinitely small differences or moments of quantities, each gives an explicit foundation for the "Method of Indivisibles" in essentially identical terms by a method which is by all relevant standards completely rigorous, being effectively equivalent to what is now known as Riemannian Integration.

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[7] As Sam Levey has pointed out to me, this will also entail that the *n* in the Archimedean Axiom would have to be allowed to range over infinite numbers. In that case, by the same reasoning as I gave in explaining Newton's proof of Lemma 1, if D is given (fixed), even if infinitely small, then we can find a quantity $c = a/n$ still smaller (and also infinitely small), provided we allow quantities to approach as close to zero as desired. But clearly such an extension of the Archimedean Axiom needs more discussion than I can give it here; see Levey's paper in this volume.

This is, of course, only a start to providing a satisfactory foundation for the infinitesimal methods used by Leibniz and Newton. In particular, it needs to be extended to the limit approach to tangents and curvature dealt with by Newton in his Lemmas, and also to issues surrounding higher-order infinitesimals. It is in fact possible to give a successful account of second-order infinitesimals on Leibniz's syncategorematic interpretation, as I have argued elsewhere (Arthur 2006a). But this beginning must suffice for present purposes.

Acknowledgements

[to be added in the proofs].

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