The transcendentality of π and Leibniz's philosophy of mathematics

Although mathematicians had believed for centuries that π must be irrational, this was not proved until 1761, by J. H. Lambert. But doubts about the rigor of his proof left the result in question until a new more rigorous proof was achieved by the great French mathematician Adrien-Marie Legendre in his *Élémens de géométrie* of 1794. After giving it, Legendre made the following remark:

> It is probable that the number π is not even contained among the algebraic irrationalities, i.e. that it cannot be the root of an algebraic equation with a finite number of terms, whose coefficients are rational. But it seems to be very difficult to prove this strictly.[1]

That is, if we call *algebraic* any number, real or complex, that is the root of an algebraic equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0, (a_i \text{ rational}),$$

then some irrationals, like √2 and 4 – √3i, are algebraic, and some, like e, are not. Such non-algebraic irrationals are called *transcendental*. Thus Legendre's conjecture just quoted is that π is such a transcendental number. The distinction itself, as Morris Kline notes, "was recognized by Euler at least as early as 1744"[2] when he conjectured that $\log_a b$ (with a, b both rational) must be either rational or transcendental. But the existence of transcendental numbers was not proved until 1840, by Joseph Liouville, and, despite the efforts of Legendre, Liouville and Hermite (who proved that e is transcendental), the proof of the transcendentality of π was not achieved until 1882, by Ferdinand von Lindemann.

So it is of some interest to see Leibniz making what seems to be the same conjecture as Legendre—that π is not expressible as the root of an algebraic equation—in an unpublished paper of 1676, a century before even its irrationality had been firmly established. My aim here is to draw attention to this curiosity, as well as to give some account of the reasoning which led Leibniz to make the conjecture. I cannot pretend, unfortunately, to give a comprehensive account of how it relates to this work in quadratures, which I believe is necessary for a full understanding; this will have to wait for another occasion.

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[1] Quoted from David Blatner, *The Joy of Pi*, London: Penguin 1997, p. 85.

[2] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, vol. 2, Oxford: OUP 1972, p. 594.

The paper in which Leibniz makes the conjecture is an unpublished manuscript, "Infinite Numbers", penned at the beginning of April, 1676, towards the end of his four years in Paris.[3] Although it contains perhaps rather too much mathematics for most historians of philosophy, its very interesting philosophical content led the editors of the German Academy edition to classify it among Leibniz's philosophical, rather than his mathematical, manuscripts. Perhaps this explains why it did not come to the notice of such mathematicians as Joseph Hofmann, who wrote the definitive account of Leibniz's mathematical inventions in his Paris years. Still, it seems to be of some historical significance, not only for the conjecture itself, but also because Leibniz makes it in the course of an analysis of the nature of magnitude that leads him to conclude that not only infinitesimals, but mathematical figures such as circles and hyperbolae, are mere fictions or *enuntiationum compendia* (abbreviations for expressions) —some 25 years earlier than he is supposed to have come to this position.

The paper begins and ends with considerations involving infinite number (thus explaining the title supplied it by the editors), although inbetween Leibniz manages to discuss an impressively diverse set of subjects: impossibility proofs of squaring the circle; the fictional status of the circle, viewed as a limit of a sequence of polygons; whether angles of contact are actual infinitesimals; a new definition of angle as a fictitious limiting entity; how the mind can sense a perfect circle if there is no image in it of one; how some things can be so small that, although we sense them in our consciousness, we cannot remember them; whether there can be a midpoint in an unbounded universe, how unbounded material things would have to move discontinuously, or rather, could not be perfectly considered without considering "the mind in them", whether infinite series can be said to have an infinitieth term; a reconsideration of the ratio of incommensurable lines, leading to a redefinition of magnitude; the definition of the sum of a converging infinite series; and finally, the thesis that an infinity of things is not one whole.

As one might have guessed, these subjects were not as heterogeneous in Leibniz's mind as they are to a twenty-first century understanding. A clue to some of the connections is provided by certain remarks of his in another unpublished paper written two months previously ("Secrets of the Sublime" of February 1676)[4] in which he was actively debating the ontic status of infinitesimals. For there Leibniz had proposed that matter's being actually divisible into points or actual infinitesimals entailed that any part would be

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[3] A VI iii n69: 495-504. In my referencing in this paper, 'A' denotes the Akademie edition of Leibniz's works, *Gottfried Wilhelm Leibniz: Sämtliche Schriften und Briefe*, ed. Deutsche Akademie der Wissenschaften (Darmstadt and Berlin: Akademie-Verlag, 1923-), followed by series and piece number, then page number. All translations are from my *Leibniz: The Labyrinth of the Continuum,* forthcoming in Yale University Press, 2000.

[4] A VI iii n60: 472-477.

commensurable with any other part; and that this in turn would appear to entail the squarability of the circle, "if it exists" (*ibid*, p. 474). If I am right, this reasoning constitutes the premise of the investigation he undertakes in "Infinite Numbers". Let me try to make it more explicit.

Suppose there are actual infinitesimals. Here the primary sense of 'actual' is that of a real as opposed to potential part of the continuum; in this sense, a finite line would be composed of an infinite number of infinitesimal linelets. This fits with Leibniz's previous analysis of the continuum in his *New Physical Hypothesis* (1671), as well as his definition of magnitude at that time as "the multiplicity of parts"[5] of a thing. However, since that time, he had had occasion to emend his definition of magnitude, noting that this definition is "worthless, unless it is established that the parts are equal to one another, or in a given ratio" .[6] This applies equally to infinitesimal parts: it's no good positing that the hypoteneuse and sides of a right-angled isoceles triangle are composed of infinitesimal lines unless the infinitesimals in each case are in the ratio of √2 to 1 (if all infinite numbers are considered equal) or equal (in which case the infinite numbers will be in the ratio of √2 to 1). This fact will be of relevance in what follows.

Now if such an interpretation can be made to work, then infinitesimals could be "actual" in a stronger sense, that is, parts of actually existing continuous wholes, not just mathematical entities. A circular atom, for example, could exist, if there were a definite (but infinite) number of infinitesimals across its diameter, and another definite but infinite number of infinitesimals around its circumference. But then the circumference and the diameter would be in a definite ratio: they would be *commensurable* in the sense of having a common measure, the assumed infinitesimal. Hence Leibniz's remark that, if "matter is actually divided into an infinity of points", it will follow that "any part of matter is commensurable with any other." In this connection, he continues, "I should examine the line of reasoning I have used elsewhere, according to which it seems to follow that a circle, if it exists, has a ratio to the diameter as one number to another".[7] This, on my reconstruction, is precisely what Leibniz is about in "Infinite Numbers", to the consideration of which I now turn.

That paper begins with Leibniz clarifying some key terms, such as 'homogeneous', and 'commensurable', applied to infinite numbers and lines. Two homogeneous infinite lines *AB...* and *CD...*, are said to be commensurable if they have a finite ratio, represented by the ratio between the finite commensurable lines *LM* and *NP* (*op. cit.*, p. 496). He then

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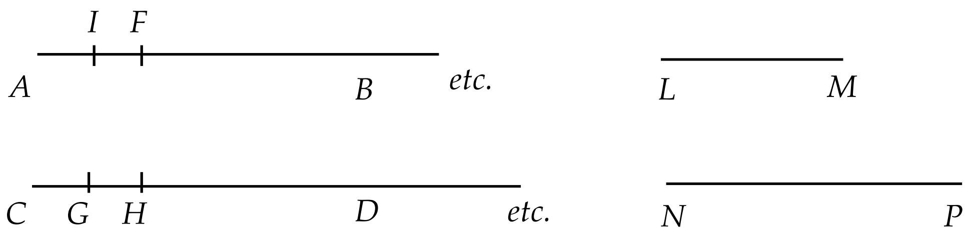
[5] See his "On the Nature of Corporeal Things" of 1671: "*Magnitude* is the multiplicity of parts" (A VI ii n452).

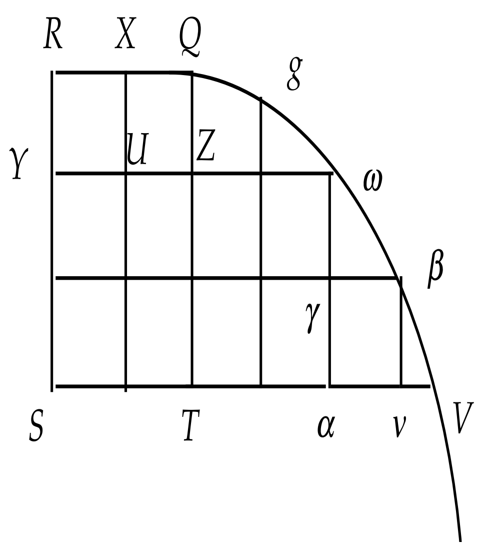
[6] See his remarks in "On Magnitude" (A VI iii n.64: 482).

[7] "On the Secrets of the Sublime" (A VI iii n60: 473).

proceeds in the next paragraph to make use of this idea of commensurable infinite lines to sketch a demonstration of the quadrature of the circle, given certain assumptions.



Suppose, he says, we have two figures, "a rectilinear one *QRST*, the other a mixed curvilinear one *QTVQ*, of the same height *QT*." Both figures are now conceived as polygons, in accordance with Leibniz's dictum that every curve is expressible as an infinitangular polygon. The areas under the curve are now "divided into an infinity of infinitely small squares" of equal size, such as $\alpha\nu\beta\gamma$ in the curvilinear figure, and *XQZU* in the rectilinear figure. Leibniz drew his axes the other way round from us, so that the height *QT* is the abscissa, and the ordinates *ZY* and $Z\omega$ are from it to the left and right, respectively.



Now comes an assumption about the "equability" of the equation of the curve:

> let us assume any arbitrary curvilinear ordinate, such as $Z\omega$, to be in rational proportion to the corresponding rectilinear ordinate, *ZY*, which will be possible if the equation of the curve allows it; then the number of infinitely small squares of one figure will also be commensurable with the number of squares of the other, and if one figure is exhausted by a repetition of squares, then so will the other be. (497)

That is, if the areas ("whole figures") are exhausted by the squares, they "will also have a common measure, namely the assumed square". But since, by hypothesis, the ordinates are in rational proportion, and the abscissa is common to both, the areas will be commensurable:

> If all these ordinates are extended in a straight line, i.e. the squares are taken directly, the number of them in the rectilinear figure will be to the number in the curvilinear one as one infinite commensurable line to another, and so as we showed above, as finite number to finite number. (497)

However, Leibniz claims to have shown elsewhere that "a curvilinear figure of the above kind is congruent with the circle". But this would mean that the square and circle are commensurable, a result he considers "absurd" (497).

Immediately Leibniz spots the reason for this "error": his quadrature of the circle depends not on the ordinates of each figure being equal to a finite number of finite common measures, but to an infinite number of infinitesimal common measures. Thus the ratio of each pair of lines does not have to be "as finite number to finite number", but can be "as infinite number to infinite" (497). Leibniz explains: "Two infinite numbers which are not as two finite numbers can be commensurable, namely if their greatest common measure is a finite number —for instance, if both are prime." Correspondingly, the figures can still be called "commensurable" in the extended sense of having an infinitesimal common measure, if each is an infinite prime number of infinitesimal common measures.

Thus there are two cases to be considered: circle and square could be commensurable (i) in the standard sense of possessing a common measure that is "finite and ordinary, (in which case they would be as finite number to finite number, which I believe to be completely irreconcilable with approximations)"; or (ii) in an extended sense of possessing a common measure that is "infinitely small, which I believe to be necessary" (497).

The first alternative, in our terms that π is rational, which Leibniz had already told Oldenburg was "probably impossible" in 1674,[8] is here dismissed as "completely irreconcilable with approximations". This undoubtedly refers to the approximations of $4/\pi$ by Wallis (in terms of an infinite product) and Brouncker (in terms of an infinite continued fraction).[9]

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[8] See Joseph E. Hofmann, *Leibniz in Paris: 1672-1676*, (Cambridge: CUP, 1974), p. 95; Leibniz to Oldenburg, 16 October 1674.

[9] See Hofmann, *op. cit.,* p. 53. Leibniz was also aware that although the "rational quadrature" of the hyperbola —that is, the expression of the ratio between the area under a hyperbola and that under the corresponding square as a rational fraction— had been accomplished by Mercator and Brouncker, no one had managed this for the circle.

What is of interest to us here, though, is the second alternative, where the circle and square could be "commensurable" in the sense of having an infinitesimal common measure. Leibniz has shown that, under the hypothesis of an actual infinitesimal common measure, the area under the curve is to the rectilinear area as the infinite lines representing the sums of their ordinates. As his figures show, these lines would then be in the ratio of the finite lines *LM* and *NP*. But his assumption that these lines would be as finite number to finite number is too strong: the circle would be squarable if these lines stood in the ratio of some algebraic irrational root (say, $\sqrt{2}$) to 1, like the diagonal and side of an isoceles right triangle. Thus if it could be shown that they are not "commensurable" even in this extended sense, the circle would not be squarable (at least by this method). At any rate, that is the best I can reconstruct Leibniz's reasoning. His own words are as follows:

> Hence now at last there seems to be a way open for a marvellous demonstration that it is impossible for there to be a quadrature of the circle of the kind we are seeking: namely one which would express the relation by an equable equation. And in order for this to be done, it must be shown that the diameter and the side [I presume he means diameter and circumference, or circumference and side] do not have even an infinitely small common measure, even in the kind of line which is as an irrational root, whether quadratic or cubic —e.g. the side of a double cube [$\sqrt[3]{2}$]— or of some higher power. (497-498)

The idea here appears to be that the infinitely small common measure involved in the kind of quadrature Leibniz is seeking would have to be a *linear* common measure. For he immediately adds: "Hence here we have a splendid use for demonstrations about incommensurables of linear [quantities], for they can also be carried over to the infinitely small, which those of arithmetic cannot." (498) Therefore, Leibniz concludes, one can prove the impossibility of squaring a circle in this sense of "expressing the relation [between circle and square] by an equable equation" if one can show that "diameter and side" do not have even (what amounts to) an algebraic irrational root. And "supposing this, it follows that the magnitude of a circle cannot be expressed by an equation of any degree." (498) In other words, π will be transcendental.

Such a proof, however, would only be as strong as its foundations. Here it depends on the idea of an ordinate y being decomposable into an infinity of infinitesimal differences, dy. As we know, Leibniz claimed that since dy is incomparably smaller than y, $y + dy$ is essentially equal to y. This, I believe, explains the following caveat, recorded by Leibniz as a marginal note to the above claims:

> This reasoning, which seems to prove that a circle is not squarable, is not to be relied upon as long as it has not been proved that the diagonal cannot —at least, by

> subtracting an infinitely small quantity— be rendered commensurable to the side, assuming an infinitely small measure. And the same holds for the other roots. (498)

Of course, Leibniz has not given the demonstration in question, and I may not have divined his reasoning accurately. It would be especially desirable to get a more certain reading of how his assumption about the "equable equation" is related to the idea of linear incommensurables. Nevertheless, Leibniz's conjecture that a proof might be possible that π cannot even be expressed as the irrational root of "an equation of any degree" seems to have anticipated by some 118 years Legendre's conjecture that it "cannot be the root of an algebraic equation with a finite number of terms, whose coefficients are rational".