‘x + dx = x’: Leibniz’s Archimedean infinitesimals

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ABSTRACT

Leibniz’s theory of infinitesimals has often been charged with inconsistency. For example, characteristic formulas such as ‘x + dx = x’ appear to be in straightforward violation of the law of identity. In this paper I offer a defence of Leibniz’s interpretation of infinitesimals as fictions, arguing that with it Leibniz provides a sound foundation for his differential and integral calculus. In contrast to some recent theories of infinitesimals as non-Archimedean entities, Leibniz’s interpretation was fully in accord with the Archimedean Axiom: infinitesimals are fictions, whose treatment as entities incomparably smaller than finite quantities is justifiable wholly in terms of variable finite quantities that can be taken as small as desired, i.e. syncategorematically. In this paper I explain this syncategorematic interpretation, showing how Leibniz used it to justify the calculus, as well as exploring its wider historical context and philosophical implications.

1. INTRODUCTION

It is a standard objection to the use of infinitesimals in calculus that their proponents want to have it both ways: on the one hand, since infinitesimals are not equal to zero, they can, unlike null quantities, have a finite ratio; but on the other hand, being incomparably small relative to a finite quantity, they can be taken to be equal to zero.[1] This was how George Berkeley objected to Newton's "moments", which Newton denoted by the letter 'o': "I admit that ... in the original notation x + o, o might have signified either an increment or nothing. But then which of these soever you make it signify, you must argue consistently with such its signification, and not proceed upon a double meaning." (*The Analyst*, Berkeley 1951, IV, pp. 74-75). John Earman (1975, p. 244) makes the same point about Leibniz's differences *dx*. He gives the example of Leibniz's derivation of the tangent to the curve

$$y = x^2/a, \qquad (1)$$

in which Leibniz proceeds by considering the point ($y + dy$, $x + dx$) infinitesimally close to the point (y, x), thus arriving at the formula

$$dy/dx = (2x + dx)/a \qquad (2)$$

Here the ratio dy/dx is defined because neither dy nor dx is strictly zero. Now Leibniz sets dx/a equal to zero on the grounds that, relative to the finite quantity x, dx is "incomparably small". This gives

$$dy/dx = 2x/a. \qquad (3)$$

But now this equation must be interpreted as an exact equality, not as an "equality up to an infinitesimal quantity" (Earman, p. 245). To put the matter succinctly, a Leibnizian "equation" such as

$$x + dx = x \qquad (4)$$

taken as a literal equation, has only the solution

$$dx = 0 \qquad (5)$$

This justifies the move from (2) to (3), but is inconsistent with the interpretation of (2), in which neither dy nor dx is strictly zero. It is true that the error in equating (2) with (3) can be

[1] Cf. the discussion of John Earman (1975), p. 244: When $\Delta x = 0$ the slope of the tangent, $\Delta y/\Delta x$, becomes equal to 0/0, an unacceptable result. "Infinitesimals were seen as a way out. Being different from 0, they allow one to avoid the 0/0 ratio, but being incomparably small, they can be neglected."

"made smaller than any assignable", as Leibniz asserts, but, Earman insists, "a contradiction is a contradiction even if it is only a small one" (245).[2]

Gilles Deleuze gives a more subtle interpretation, coining the term "vice-diction" to describe such vanishingly small contradictions (46). He argues that "difference implies the negative, and allows itself to lead to contradiction only to the extent that its subordination to the identical is maintained" (xix). He correctly sees that Leibniz justifies his treatment of differentials by reference to his Law of Continuity. According to this principle, a parabola can be regarded as the limiting case of an ellipse as the second focus is removed from the first, or rest the limiting case of motion as it becomes gradually slower, or a point as the limit of a line as the line is reduced in magnitude, even though in each example the series of cases or species (ellipses, motions, lines) terminates in "an opposing quasi-species" (46), (parabola, rest, point).[3] "Likewise equality can be considered as an infinitely small inequality" (L 352). "And although these terminations are excluded," Leibniz explains, "that is, are not included in any rigorous sense in the variables which they limit, they nevertheless have the same properties as if they were included in the series, in accordance with the language of infinites and infinitesimals." (L 546) The neologism "vice-diction" is intended to capture this state of affairs. Although the parabola is essentially different from the ellipse, excluding it "*in essence*", it includes it as a limiting *case*: "It does not contain the other in essence, but only with respect to properties, in *cases*" (46). Thus "in the infinitely small, ... the unequal vice-dicts the equal, and vice-dicts itself, to the extent that it includes in the case what it excludes in essence" (46).

But how does this resolve anything? Is there still a contradiction at the level of properties or cases? "In reality," Deleuze admits, "the expression 'infinitely small difference' does indeed indicate that the difference vanishes so far as intuition is concerned." (46) "It matters little whether the supposed negative of difference is understood as a vice-dicting limitation or a contradicting limitation [as in Hegel]" (50). What is at issue, according to Deleuze, is the alternative between infinite and finite representation (176):

> That is why the metaphysical question was announced from the outset: why is it that, from a technical point of view, the differentials are negligible and must disappear in the result? It is obvious that to invoke here the infinitely small, and the infinitely small

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3 "a given ellipse approaches a parabola as much as is wished, so that the difference between" Loemker p. 352)

> magnitude of the error (if there is ‘error’), is completely lacking in sense and prejudges infinite representation. The rigorous response was given by Carnot in his famous *Reflections on the Metaphysics of the Infinitesimal Calculus*, but precisely from the point of view of a finite interpretation ... (177)

Thus Deleuze counterposes the “infinite representation” he sees as underpinning Leibniz’s viewpoint —one “completely lacking in sense”— with the finitist interpretation of the calculus urged by Carnot, one that has now become the norm. On the finitist interpretation the “fate of the calculus” is no longer tied to infinitesimals:

> We know in effect that [1] the notion of limit has lost its phoronomic character and involves only static considerations; that [2] variability has ceased to represent a progression through all the values of an interval and come to mean only the disjunctive assumption of one value within that interval; that [3] the derivative and the integral have become ordinal rather than quantitative concepts; and finally that [4] the differential designates only a magnitude left undetermined so that it can be made smaller than a given number as required. (176; numeration added)

“The birth of structuralism at this point,” he adds, “coincides with the death of any genetic or dynamic ambitions of the calculus” (176).

Now, as a statement of the received view of the status of Leibniz’s infinitesimals, I think this appraisal cannot be faulted. But *I do not think it is a correct interpretation of Leibniz’s own position on infinitesimals*; and, what is interesting, given Deleuze’s insightful remark about the connection of modern analysis with the birth of structuralism, is that Leibniz himself upheld [4], the Archimedean interpretation of infinitesimals, while upholding the phoronomic character of the limit, the conception of physical quantities as continuously varying, and the quantitative nature of the derivative and integral, in opposition to [1]-[3]. In what follows, I shall give my interpretation of Leibniz’s foundation for his calculus, showing how it is not wanting in rigour, even though the rigour is expressed from within a dynamic and not a structural rubric. Thus when Deleuze says there is a “treasure buried within the so-called barbaric or pre-scientific interpretations of the differential calculus” (170), I agree, even if I do not believe he has correctly identified where the treasure lies. If I am right about the foundation Leibniz gives his calculus, and Deleuze is correct in his analysis of the connection between the modern foundation and structuralism, then Leibniz’s own interpretation of the calculus contains the seeds of an alternative to structuralism.[4]

2. LEIBNIZ’S SYNCATEGOREMATIC INTERPRETATION

Leibniz’s commitment to the actual infinite has been much misunderstood. For he would often uphold the actual infinite while denying infinite number in the same breath, as in this passage in the *New Essays*:

> It is perfectly correct to say that there is an infinity of things, i.e. that there are always more of them than can be specified. But it is easy to demonstrate that there is no infinite number, nor any infinite line or other infinite quantity, if these are taken to be genuine wholes. (Leibniz 1704/1981, 157, Russell 1900, 244)

For this reason he has often been accused of inconsistency by Cantorians, beginning with Cantor himself:

> Even though I have ... quoted many places in Leibniz’s works where he comes out against infinite numbers, ... I am still, on the other hand, in the happy position of being able to cite pronouncements by the same thinker in which, to some extent in contradiction with himself, he expresses himself unequivocally *for* the actual infinite (as distinct from the Absolute).[5]

Nicholas Rescher and Gregory Brown (among others) have taken a similar stand: Leibniz was ahead of his time in realizing the need for the actual infinite, in opposition to the traditional Aristotelian opposition to anything but the potential infinite. But his stubborn attachment to the Part-Whole Axiom prevented him from embracing infinite number and anticipating Cantor in the process.[6]

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[4] I will not explore the implications for an alternative to structuralism beyond this setting straight of the Leibnizian foundations, even though there is much to be said — especially about the history of the interpretation of these foundations: their appropriation by Hermann Cohen and the neo-Kantians, and also by Bergson, and how some of these authors’ misconceptions paved the way for the victory of Cantor, Russell and the set-theoretic foundations that have come to dominate mathematics. D264

[5] Georg Cantor, “Grundlagen einer allgemeinen Mannigfaltigkeitslehre” (1883), in Cantor (1932, 179).

[6] Rescher states that Cantor’s theory of transfinite numbers, point-set topology and measure theory “have shown that Leibniz’s method of attack was poor. Indeed, Galileo had already handled the problem more satisfactorily ...” (Rescher 1967, 111); Gregory Brown charges Leibniz with a failure of nerve with respect to infinite number (Brown 1998, 122-123; 2000, 23-24).

Lately, however, the subtlety of Leibniz’s position on the infinite has begun to be appreciated, and it has become clear that his mature account of the actual infinite is actually perfectly consistent (see Arthur 1999, 2001a, b, Levey 2008). According to this account the term ‘infinite’ does not refer to an entity, even though it makes perfectly good sense in appropriate contexts; it is, to use the terminology of the Schools, a *syncategorematic term*. Thus Leibniz holds that although there is an actual infinity of the parts into which matter can be divided (to use his leading example), this does not mean that there is a number “infinity” that can be assigned to them. Any piece of matter is actually —not merely potentially— divided into further parts, but there is no totality or collection of all these parts. His argument for this infinite division is given very clearly in a manuscript written ca. 1680:

> Created things are actually infinite. For any body whatever is actually divided into several parts, since any body whatever is acted upon by other bodies. And any part whatever of a body is a body by the very definition of body. So bodies are actually infinite, i.e. more bodies can be found than there are unities in any given number. (A VI iv 1393; A235).

From this we see that to assert the infinitude of these parts is to assert that there are more of them than can be assigned by any number finite N. Nevertheless, as he makes clear in other texts from the same period, there is no number that is greater than all N. In terms of predicate calculus, we may express this as follows:

- To assert an infinity of things *syncategorematically* is to say that for any finite number x that you choose to number the things, there is a number of things y greater than this: $\forall x\ \exists y\ (Fx \rightarrow y > x)$, with Fx = x is finite, and x and y numbers.

This is contrasted with the *categorematic* sense of infinity, according to which to say that there are infinitely many parts is to say that there is a number of parts greater than any finite number, i.e. that there is an infinite number of parts:[7]

- To assert an infinity of things *categorematically* would be to assert that there exists some one number of things y which is greater than any finite number x, i.e. that $\exists y\ \forall x\ (Fx \rightarrow y > x)$.

The first formula does not commit you to infinite number, since it merely asserts that there is a greater number y than the one you chose, and y may be finite. But the categorematic

[7] For a brief but enlightening discussion of the categorematic/syncategorematic distinction, see A. W. Moore (1990), 51-52.

expression commits you to the existence of a number greater than all finite numbers; and in agreement with Ockham, Leibniz denies that there is such a number as infinity: an infinite number is not an entity, and nor is any infinite whole, in agreement with what he wrote in the passage from the *New Essays* quoted above.[8]

Admittedly, Cantorian prejudices are so strong that many today have trouble seeing the cogency of this position. If there are actually infinitely many parts of matter at any given time, why wouldn't there be an actually infinite number of them? Why wouldn't their cardinality be $\aleph_0$, aleph null? Isn't the denial of infinite number the traditional position, where the only infinite is the potential infinite? On that Aristotelian view, to say that there are infinitely many is just to say that no matter how many one takes, there *could* be more, but there are always finitely many. Let me attempt to explain the difference between the syncategorematic conception of the actual infinite and the other two conceptions by reference to the example of the *infinitude of the primes*. This was first proved in Euclid's *Elements* in one of the earliest and most beautiful examples of reasoning by *reductio ad absurdum*. As is well known, Euclid proceeds as follows.

He supposes that there is a greatest prime, i.e. that there exists some finite prime such that all primes are less than or equal to it: $\exists x \forall y(Fx \;\&\; y \leq x)$, where Fx = x is finite, and x and y range over prime numbers. He then constructs a number as follows: take the supposedly greatest prime, and multiply it successively by each prime smaller than it, then add 1 to the result. This is now a number which is greater than all the primes, but which is not divisible evenly by any of the primes; it is therefore itself a prime, and greater than the supposedly greatest prime, thus contradicting the starting supposition. Therefore (since this construction is completely general, and could be performed on any supposed greatest prime), it follows by *reductio ad absurdum* that the supposition was false, so that $\neg\exists x\forall y(Fx \;\&\; y \leq x)$. Turning the cranks of predicate logic, the latter expression is equivalent to $\forall x\exists y\neg(Fx \;\&\; y \leq x)$, and this in turn to $\forall x\exists y(Fx \rightarrow y > x)$ —*for every finite prime there is a greater prime*.

But this last expression is the syncategorematic infinite, from which the categorematic infinite $(\exists y)(\forall x)(Fx \rightarrow y > x)$ —*there is a prime greater than every finite prime*— does not

[8] Leibniz draws some very interesting metaphysical consequences from this position on the infinitude of the parts of matter. As I have explained in more detail elsewhere (Arthur 1998, 2001a), it is because the multiplicity of parts in a given body is "a quantity which is in fact greater than any given number for any aggregate that is sensible or corresponds to phenomena" (G. II. 282; to De Volder) that bodies, taken as wholes, *are* mere phenomena. The parts are actual, the whole, being infinite, is not.

follow. To think that by asserting the infinitude of the primes one thereby commits oneself to the categorematic infinite is to commit the Quantifier Shift Fallacy!

Of course, Cantor himself does not commit this fallacy. He would agree that there is no more a greatest prime than a greatest natural number: ω, the first infinite (transfinite ordinal) number, comes after all the natural numbers, and therefore after all the primes. Of course, he would still say that ω and $\aleph_0$ (the first transfinite cardinal) are actually infinite, categorematic, numbers. My point, though, is that this is an *extra assumption*; it is not necessary to assume that there is an actually infinite *number* in order to assert an actual infinity of primes, if this infinity is understood syncategorematically.

The contrast with the potential infinite is perhaps equally subtle. Aristotle himself declared that the mathematicians need only the potential infinite. For one could understand Euclid’s proof to have proved only that no matter how great a prime number one takes *it is always possible* to create a greater one. Now, the latter statement is true; but the *possibility* in question relates to the construction. In the context of the *reductio* proof, though, what the construction establishes is that, given any finite prime, there actually is a prime greater than it. The possibility of the construction establishes that *there are actually infinitely many primes*.

Turning now to infinitesimals, it has also long been held that Leibniz did not have a very coherent position here either. His attempts to explain them as *fictions* has been seen as something of a piece of diplomacy in aid of his supporters in France such as L’Hôpital, a last ditch effort to save them from the criticisms of Rolle and Nieuwentijt. Again, however, the subtlety of Leibniz’s position has been underestimated, as Hidé Ishiguro has argued in the second edition of her *Leibniz’s Philosophy of Logic and Language* (1990). There she has given an insightful presentation of Leibniz’s mature interpretation of infinitesimals which is completely in accord with the interpretation of his views on the infinite that I have presented above. According to Ishiguro, Leibniz’s position is analogous to Russell’s regarding definite descriptions: “one can have a rigorous language of infinity and infinitesimal while at the same time considering these expressions as being syncategorematic (in the sense of the Scholastics), i.e. regarding the words as not designating entities but as being well defined in the proposition in which they occur” (82).

This is contrary to the usual understanding (faithfully recounted by Deleuze), where Leibniz is understood as committed to an “infinite representation”. Even Henk Bos (1974/75, 54-56), whose profound contribution to the understanding of Leibniz’s differential calculus we will depend on later in this paper, takes Leibniz to have provided two different approaches to interpreting infinitesimals. One is finitist and Archimedean, in which differentials are interpreted as finite differences that may be taken so small as to lead to an error less than any assignable. The second is based on the Law of Continuity: it accepts infinitely small quantities as “true quantities of their own sort”, but insists on interpreting them as fictions. But as Ishiguro has argued, these approaches are in fact two sides of the same coin. To say that *dx* is a fiction is *not* to say that there exist “fixed entities with non-Archimedean magnitudes, the introduction of which shortens proofs” (Ishiguro 1990, 83). “The word infinitesimal does not designate a special kind of magnitude. It does not designate at all.” (83) This is what is meant by calling the interpretation *syncategorematic*: terms involving infinitesimals are “ostensibly designating expressions which follow certain *sui generis* rules” (83) and whose introduction shortens proofs; but they do not in fact designate real entities. The syncategorematic interpretation explains how it is possible to treat infinitesimals *as if* they are infinitely small actuals under certain well-defined conditions. As Ishiguro puts it,

> When we make reference to infinitesimals in a proposition, we are not designating a fixed magnitude incomparably smaller than our ordinary magnitudes. Leibniz is saying that whatever small magnitude an opponent may present, one can assert the existence of a smaller magnitude. (87)

As she goes on to argue, “Leibniz denied that infinitesimals were fixed magnitudes, and claimed that [in our apparent references to them] we were asserting the existence of variable finite magnitudes that we could choose as small as we wished” (92). Leibniz’s infinitesimals, that is, are in keeping with the Archimedean Axiom.

In what follows I shall argue for the correctness of Ishiguro’s interpretation by reference to Leibniz’s own writings on foundations. I will argue that Leibniz’s characterization of infinitesimals as fictions is not a stratagem invented to save his embarrassment at the fact that the calculus of infinitesimals had worked so well despite a supposed lack of foundation. Rather, it has a precise mathematical content, perfectly consistent with his philosophy of the infinite and solution to the continuum problem (Arthur 2001, 2006b). Moreover, I shall argue here, this content is given by the foundation of the method on the Archimedean Axiom.

3. Leibniz's Justification of the Calculus

The Archimedean Axiom (actually due to Eudoxus) asserts that for any two geometric quantities x and y (with $y > x$), a natural number n can be found such that $nx > y$. This entails the corollary that no matter how small a geometric quantity is given, a smaller can be found:

> **Axiom of Archimedes**: "Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another" (Euclid, *Elements*, Book V, Def. 4). That is, for any two geometric quantities x and y (with $y > x$), a natural number n can be found such that $nx > y$.
>
> **Corollary**: no matter how small a geometric quantity is given, a smaller can be found.

It is this corollary to Archimedes' axiom that precludes actual infinitesimals, where an *actual infinitesimal* may be thought to be the counterpart of the *categorematic infinite*. Just as the categorematic infinite is a number greater than all finite numbers, so an actual infinitesimal quantity may be defined as a quantity smaller than any finite quantity.

> **Actual Infinitesimal**: a quantity smaller than any finite quantity.

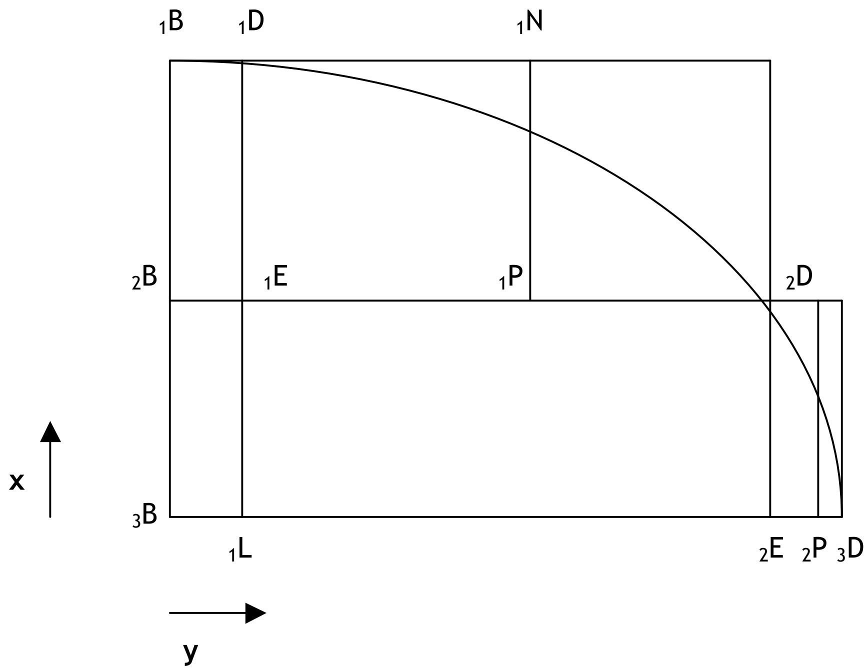
This explains why actual infinitesimals are called "non-Archimedean quantities", and a geometry incorporating them a non-Archimedean system. (I have defined all these items for geometry; similar definitions can be given for numbers, so that an actual or non-Archimedean infinitesimal is a number smaller than any finite number.)

It is a mark of the huge influence of Bertrand Russell on twentieth century philosophy of mathematics that it is widely accepted that non-Archimedean infinitesimals were "driven out of mathematics once and for all, or so it seemed" by Cantor and Dedekind before they were "rehabilitated as perfectly good numbers" by the nonstandard models of analysis invented by Abraham Robinson in the 1970s. As a matter of historical fact, however, despite the near total banishment of infinitesimals from the calculus, a consistent algebraic theory of non-Archimedean infinitesimal magnitudes was developed already in the 1870s, and was developed in the early decades of the twentieth century into theories of non-Archimedean ordered groups and semigroups, non-Archimedean ordered algebraic systems, and non-Archimedean geometry. So it was not at all the case that mathematics had to wait for Robinson to rehabilitate infinitesimals.

Ironically, Leibniz himself entertained three more or less distinguishable theories (or theory-sketches) of actual infinitesimals prior to his invention of the calculus that bear several points of analogy with some of the modern approaches (see Arthur 2007). What compounds this irony quite deliciously, however, is the additional fact that in the very work in which he first laid out his infinitesimal methods for solving quadratures Leibniz gave a well thought-out justification for these infinitesimals on a thoroughly Archimedean foundation!

The work in question is a comprehensive Latin treatise, *De quadratura arithmetica*, written by Leibniz in 1675-76, which has only recently been edited and published by Eberhard Knobloch (Leibniz 1993). In this treatise, as Knobloch has shown, “Leibniz laid the rigorous foundation of the theory of infinitely small and infinite quantities” (2002, 59); and Knobloch’s interpretation of Leibniz’s foundational work, as I have argued elsewhere (Arthur 2006a), is fully in keeping with Hidé Ishiguro’s attribution to Leibniz of a “syncategorematic” interpretation of infinitesimals. In *De quadratura* Leibniz promotes his new method of performing quadratures directly “without a *reductio ad absurdum*” (Prop 7, *Scholium*; *De quadratura*, 1999, p. 35), by what we would now call a direct integral. This, he believes, necessarily involves the assumption of “fictitious quantities, namely the infinite and the infinitely small” (35). The traditional Archimedean method of demonstration was by a double *reductio ad absurdum*: it would be shown that a contradiction could be derived on the assumption that the quantity in question was smaller than a given value, and another contradiction on the assumption that the quantity in question was greater than that value, thus proving that it equalled it. Leibniz’s method is instead to proceed by an application of the Archimedean Axiom, appealing to the corollary that no matter how small a geometric quantity is given, a smaller can be found. Thus he prefers a justification “which simply shows that the difference between two quantities is nothing, so that they are then equal (whereas it is otherwise usually proved by a double *reductio* that one is neither greater nor smaller than the other)” (35). That is, he applies the Archimedean axiom to demonstrate that the error involved in calculations with infinitely small differences can be reduced to a quantity less than any given quantity by taking a difference sufficiently small, rendering it effectively null.

Moreover, this justification does not have to be effected in every case: it is enough to show that it can be done in a general case. This Leibniz does in a case that is surprisingly general, given the usual accusations about the parlous lack of justification he and Newton are alleged to have provided for their methods. For the key theorem that Leibniz successfully demonstrates in *De quadratura arithmetica* using this Archimedean method is Proposition 6, a theorem that rigorously justifies what is now known as Riemannian Integration, as Eberhard Knobloch has shown in detail (2002).[9] (Leibniz provides a similar justification for his Theorems 7 and 8). The demonstration proceeds as follows (1999, pp. 30-33).



Leibniz first identifies and then relates the sum of the "elementary rectangles [*rectangula elementales*]" (that is the Riemannian sum of unequal rectangles by which the curve is being approximated, which we may denote Q), and that of the mixtilinear figure [*spatium gradiformis*] that is the area under the curve between two ordinates $_1L$ and $_3D$, which we may denote A.[10] Then he demonstrates that the difference between the area and the sum of the elementary rectangles, A – Q, can be no greater than the area of a certain rectangle whose height is the maximum height h_m of any of the elementary rectangles, and whose width is the distance between the two ordinates $_1L$ and $_3D$. Thus $A - Q \leq {}_1L_3D \times h_m$. But because the curve is assumed continuous, Archimedes' Axiom applies: "this greatest height (an abscissa) can be chosen smaller than any given quantity, because the curve is continuous"

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[9] The exposition I give is indebted to Knobloch's (2002), whose simplified diagram I reproduce, while following as closely as possible Leibniz's own symbolization from the *De quadratura* (Leibniz 1999).
[10] Here I am following Knobloch's symbolization, where $_3D$ is the greatest ordinate in place of Leibniz's $_4D$, and the greatest height of any of the elementary rectangles is h_m in place of his ψ_4D: in the original the sum of the elementary rectangles $\leq {}_1L_4D \times \psi_4D$ (Leibniz 1993, 29-32).

(65). Thus the height h_m, even though it is greater than the heights of all the other elementary rectangles, “can be assumed smaller than any assigned quantity, for however small it is assumed to be, still smaller heights could be taken.” Therefore the area of the rectangle ${}_1L_3D \times h_m$ “can also be made smaller than any given surface”. It therefore follows that the difference between the area under the curve and the Riemannian sum, A – Q, “can also be made smaller than any given quantity. QED.” (pp. 30-33). There is therefore no error involved in calculating the quadrature as the sum of an infinity of infinitesimal areas, provided this is understood to mean that there are more little finite areas than can be assigned, and that their magnitude is smaller than any that can be assigned.

It is worth recalling at this point Deleuze’s charge that “It is obvious that to invoke here the infinitely small, and the infinitely small magnitude of the error (if there is ‘error’), is completely lacking in sense and prejudges infinite representation.” This would only be so if the infinitesimal areas in question were non-Archimedean, or actual infinitesimals. In fact, however, what Leibniz has done is to invoke finite areas that can always be made smaller than any preset magnitude. He has justified proceeding, in this case, as if there were an infinity of infinitesimals precisely without assuming an infinite representation! It is also hard to see any difference in rigour between his justification and the finitist justifications of Carnot and Cauchy. Thus Leibniz appears justified in remarking about this theorem, “it serves to lay the foundations of the whole method of indivisibles in the soundest way possible” (1993, 24); Knobloch agrees, calling it a “model of mathematical rigour (2002, 72)”.

The point here is not that Leibniz has two methods, as Bos supposed, one committed to the existence of infinitesimals and the other Archimedean;[11] nor is it the case that he simply uses the infinitesimal calculus and then airily refers to the fact that one could *instead* have used an Archimedean method. It is that, as examples like this demonstrate, the Archimedean Axiom justifies proceeding as if there are infinitesimals, and at the same time demonstrates that what they really stand for are finite quantities which can be taken as small as desired. Once this is demonstrated in a suitably general case, it also justifies the use of these fictions in other analogous cases. As Leibniz himself writes, “Nor is it necessary always to use inscribed or circumscribed figures, and to infer by *reductio ad absurdum*, and to show that the error is smaller than any assignable; although what we have said in *Props. 6, 7 & 8*

[11] Cf. Bos (1974/75) on “Leibniz’s two different approaches to the foundation of the calculus” (p. 55).

establishes that it can easily be done by those means.” (Scholium to *Prop. 23*, Leibniz 1993, 69)

In effect, the application of the Archimedean Axiom enables a kind of Arithmetic of the Infinite. In his article, Knobloch identifies a number of rules which are tacitly applied by Leibniz in *De quadratura*, “without demonstrating them, only relying on the ‘law of continuity’” (2002, 67). Examples are

1. Finite + infinite = infinite.
2. 2.1 Finite ± infinitely small = finite

   2.2 If $x = y$ + infinitely small, then $x - y \approx 0$ (is unassignable), where x and y are finite quantities. ...

10. Finite divided by infinitely small = infinite divided by finite = infinite (greater than any assignable ratio).
11. Infinitely small divided by finite = finite divided by infinite = infinitely small (infinitesimal)

It will be an instructive exercise to see how these rules can be justified by the Archimedean foundation outlined above. Let us take 2.2 first. If $x = y + dy$, where dy is an arbitrarily small variable quantity, and D is any pre-assigned difference between x and y, no matter how small, then dy may always be taken so small that dy < D. In particular, if D is supposed to be some fixed ultimate difference between them, then dy can be supposed smaller: so long as D and dy are quantities obeying the Archimedean axiom, the variability of dy means that it can always take a value such that dy < D for any assigned D. Therefore, since the difference between x and y is smaller than any assignable, it is unassignable, and effectively null. The same reasoning justifies 2.1.

Leibniz gives such an argument explicitly in a short paper dated 26 March,1676, in the same time period in which he was composing *De quadratura*:

> We need to see exactly whether it can be demonstrated in quadratures that a difference is not after all infinitely small, but nothing at all. And this will be shown if it is established that a polygon can always be inflected to such a degree that even

> when the difference is assumed infinitely small, the error will be smaller. Granting this, it follows not only that the error is not infinitely small, but that it is nothing at all —since, of course, none can be assumed. (A VI iii, 434; Leibniz 2001, 64-65)

Although one might wish for more perspicuous wording, Leibniz's reasoning here seems very evocative of Newton's in his justification of the method of first and ultimate ratios:

> *Quantities, or ratios of quantities, which in a given time constantly tend to equality, and before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal*. If you deny this, let their ultimate difference be D. Then they cannot approach so close to equality that their difference is less than the given difference D, contrary to hypothesis.[12]

This consilience of the foundations of Leibniz's and Newton's justifications of quadratures is discussed more fully elsewhere (Arthur 2008). Moving on to Knobloch's Rule 1, suppose dz is another arbitrarily small finite variable quantity such that the ratio $dy:dz$ remains finite as dz is made arbitrarily small. Now again suppose $x = y + dy$, and divide all through by dz, and let dz become arbitrarily small. As it does so, x/dz and y/dz will each become arbitrarily large; indeed, no matter how large a quantity Q is given, dz can be taken sufficiently small that x/dz and y/dz will each exceed it. Thus x/dz and y/dz will each be greater than any given quantity Q,. They will therefore both be infinite in Leibniz's sense, i.e. greater than any assignable, while $dy:dz$ remains finite, yielding rule 1, Finite + infinite = infinite.

For Rule 10, let dx be an arbitrarily small quantity, and x be any finite quantity. Then no matter how large a quantity Q is given, since dx can become arbitrarily small, it can become sufficiently small that $dx < x/Q$, i.e. $x/dx > Q$. That is, x/dx is greater than any given quantity Q, and is therefore unassignable; or, finite divided by infinitely small = infinite divided by finite. As should by now be evident, similar justifications can be given for all of Knobloch's other rules.

This is, of course, only a start to providing a satisfactory foundation for the infinitesimal methods used by Leibniz and Newton. In particular, it does not treat all those issues surrounding second- and higher-order infinitesimals; although, as I have argued elsewhere

[12] Newton (1999, 433); quoted in the version of the first edition of 1687. Ironically, when Leibniz saw Newton's *Principia* for the first time he misconstrued this *reductio*, making the rather lame remark that "It can be doubted whether there is an ultimate difference" (Bertoloni Meli 1993, 226), when Newton was in fact proving that there cannot be one.

(Arthur 2008c), it is possible to give a successful account of second-order infinitesimals on Leibniz’s syncategorematic interpretation. But this must suffice for present purposes.

4. Infinitesimals and Leibniz’s Law of Continuity

In his mature attempts to ground the calculus, Leibniz appeals to his Law of Continuity, as we have already seen mentioned by Deleuze. So it remains to say something about this. Leibniz was well aware of the need to justify the transition to a limit. Indeed, as Ishiguro observes, this was the point of his Law of Continuity, which he phrased as follows in his (1701):

> If any continuous transition is proposed that finishes in a certain limiting case (*terminus*), then it is possible to formulate a general reasoning which includes that final limiting case.[13]

As it was worded in its first publication in 1688, “A Certain General Principle, Useful not only in Mathematics but in Physics”,

> *When the difference between two instances in a given series, or in whatever is presupposed, can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought, or what results, must of necessity also be diminished or become less than any given quantity whatever.* (A VI 4, 371, 2032)[14]

This principle is not very clear from these succinct formulations. As Bos observes, Leibniz explains it more clearly through his examples. A reasoning about ellipses, for instance, could be extended to parabolas by introducing a second focus infinitely distant from the first:

> … [For example,] we know that a given ellipse approaches a parabola as closely as desired, so that when the second focus of the ellipse is removed far enough away from the first focus, the difference between the ellipse and the parabola becomes less than any given difference, since then the radii from that distant focus differ from parallel lines by an amount as small as desired. (2032)

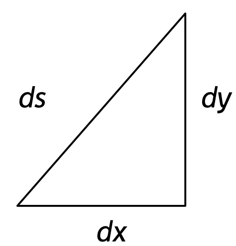
Thus provided one knows there is a number (albeit irrational) to which (in the case of the circles above) the inscribed and circumscribed polygons approach arbitrarily closely, or that

[13] “Proposito quocunque transitu in aliquem terminum desinente, liceat ratiocinationem communem instituere, qua ultimus terminus comprehendatur.” (1701, 40).

[14] Leibniz adds: “Or, to put it more commonly, *when the cases (or given quantities) continually approach one another, so that one finally passes over into the other, the consequences or events (or what is sought) must do so too.*” (A VI 4, 371, 2032)

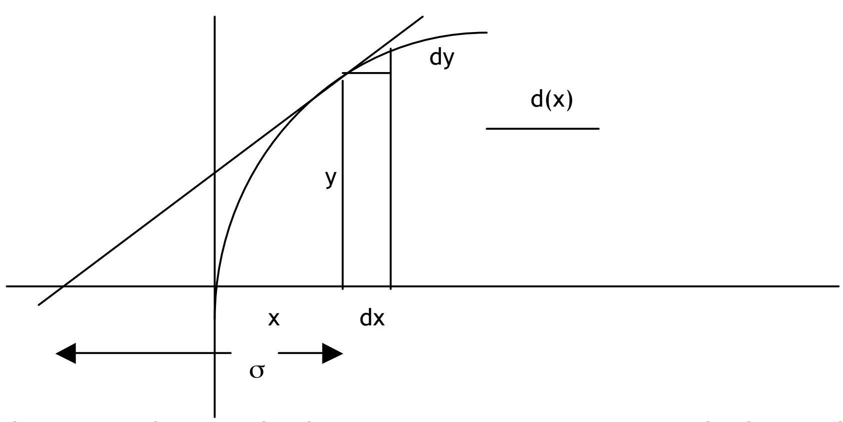
there is a figure to which the ellipse approaches arbitrarily closely as one of its foci is removed indefinitely far away, the limit —which lies as a fictional entity outside the transition itself— can be fictitiously included as the *terminus* of the transition. Such extensions, as Bos explains, "involve the use of terms or symbols which become meaningless in the limiting case, while the argument they describe remains applicable, and in such cases the terms and symbols can be kept as 'fictions'. According to Leibniz, the use of infinitesimals belongs to this kind of argument." (Bos 1974-75, 57).

In his manuscript of c.1701, *Cum prodiisset…* (first published in Gerhardt's edition 1846), Leibniz presents an explicit foundation for his infinitesimals on the basis of this Law of Continuity, as Bos has explained with admirable lucidity. It proceeds as follows. *dx* and *dy* are finite, arbitrarily small, and variable: they are neither fixed quantities, nor infinitely small ones. In the characteristic triangle,



ds can be treated as the side of an inscribed finite polygon, which can vary in length to become arbitrarily small. Now let *(d)x* be a fixed finite line segment. For all finite *dx* and *dy*, *(d)y* may now be defined by the proportion

$$(d)y:(d)x = dy:dx \qquad (1)$$



But the same *(d)y* can also be given an interpretation in the limit when the variable *dx* = 0, namely through the proportion

$$(d)y:(d)x = y:\sigma \qquad (2)$$

where σ is the subtangent to the curve.

Now, for all finite $dx \neq 0$, $(d)y:(d)x$ can be substituted for $dy:dx$ in any formula. But since the resulting formula is still interpretable even in the case where $dx = 0$, the Law of Continuity asserts that this limiting case may also be included in the general reasoning: $dy:dx$ can be substituted for $(d)y:(d)x$ in the resulting formulas even for the case where $dx = 0$, with dy and dx in this case interpreted as fictions. If a third variable v is involved, which varies with x, $(d)v:(d)x$ can be defined in an entirely analogous way.

That this foundation suffices for first-order differentials and the rules of the calculus is best shown by an example. In *Cum prodiisset…* Leibniz offers the following proof of the rule for the differentiation of a product $d(xv) = xdv + vdx$. He lets $ay = xv$ (here the purpose of the constant a is to conserve the homogeneity of the equation), and then allows x, y, and v all to increase infinitesimally. Following Bos, I quote his demonstration:

> Proof: $ay + dy = (x + dx)(v + dv)$
>
> $$= xv + xdv + vdx + dxdv,$$
>
> and, subtracting from each side the equals ay and xv, this gives
>
> $$ady = xdv + vdx + dxdv$$
>
> or $$ady/dx = xdv/dx + v + dv$$
>
> and transposing the case as far as possible to lines that never vanish, this gives
>
> $$a(d)y/(d)x = x(d)v/(d)x + v + dv$$
>
> so that the only term remaining which can vanish is dv, and in the case of vanishing differences, since $dv = 0$, this gives
>
> $$a(d)y = x(d)v + v(d)x$$
>
> as was asserted. … Whence also, because $(d)y:(d)x$ always $= dy:dx$ one may assume this in the case of vanishing dy, dx and put
>
> $$ady = xdv + vdx.$$ [15]

This approach to securing the foundations of the calculus is clearly very similar to the finitist version for which Cauchy became famous. As Bos argues, it leads very naturally to the

[15] Leibniz, "Cum prodiisset…", pp. 46-47; quoted from Bos (1974-75, 58). The ellipsis omits an obvious error in calculation not important for the argument.

concept of the function, and even to the introduction (later in the history of the calculus) of the concept of derivative. Indeed, Leibniz's definition in (1) of *(d)y* as (dy/dx) × *(d)x* , coupled with the stipulation in (2) that when $dx = 0$ the secant becomes the tangent, implies that *(d)y* is identical to the differential of Cauchy, defined as $f'(x)\,(d)x$. For Cauchy defines $f'(x)$ as the limit of $\{f(x + \Delta x) - f(x)\}/\Delta x$ as $\Delta x \to 0$. The approach leads (through Euler and others) to the idea of differential quotients, but crucially depends on the recognition that one has to choose one of the variables as independent. Even though Leibniz's difficulties and mistakes in the dispute about second order differentials with Nieuwentijt stems from his confusion on this, as evidenced also in his own attempt to extend the above type of reasoning to justify second order differentials, Bos shows that a correct justification for second order differentials can be given along these lines. It is just that they will depend on a certain choice of the progression of the variables, i.e. on a choice of which differential is taken to be constant.

Returning to the criticisms of Leibniz's (and Newton's) infinitesimals with which we began, it should now be clear that Earman's allegation of a contradiction is not warranted. The above proof, for example, of the rule for the differentiation of a product $d(xv) = xdv + vdx$, does not depend on an interpretation of *dv* or *dx* as actual infinitesimals; nor are they fixed finite terms. Leibniz's differentials are finite, arbitrarily small, and variable. And given the Archimedean foundation of his methods, this is enough to ensure the rigour of proofs involving them; as well as to justify treating them as if they are infinitely small, and an infinity of them can sum to a finite quantity.

And as for Deleuze's criticisms, we see that Leibniz's foundation, in falling between the modern Weierstaßian "static" or "structural" analysis, on the one hand, and a self-contradictory "infinite representation", on the other, does not relinquish its phoronomic character. Leibniz's analysis is developed to apply to continuous transitions, and the Law of Continuity is an extension of the Archimedean Axiom to explicate such cases. Leibniz's justification of infinitesimals is (in its own terms) a valid one.

5. Leibniz's Philosophy of Difference

In closing, I would like to make some alluring remarks about the connection between Leibniz's use of the Axiom of Archimedes in his mathematics and his more general philosophical principles.

A couple of years or so after he wrote the *De Quadratura* which we were considering above, Leibniz makes mention (in an unpublished manuscript) of what he calls his “Herculean argument”:

Herculean argument (1678/9?):
“But here there is a place for that Herculean argument of mine, All those things which are such that it is impossible for anyone to perceive whether they exist or not, are nonexistent.”

As I point out in my editorial notes, this principle might well be dubbed Leibniz’s “Principle of the Nonexistence of Imperceptibles.” What is interesting is that it points up a profound link between the Archimedean Axiom in mathematics and Leibniz’s more famous Principle of the Identity of Indiscernibles, since both may be derived from it. To see this we should first apply it to *differences* in general. This gives us:

Principle of the Nonexistence of Imperceptible Differences:

Any difference between two things which is in principle imperceptible is nonexistent.

Therefore, since indiscernible things are those whose difference is in principle imperceptible, and identical things are those whose difference is nonexistent, we have

Principle of the Identity of Indiscernibles:

Any two things that are indiscernible are identical.

And if the principle of the non-existence of imperceptible differences is applied in mathematics, where Leibniz habitually equated the given or assignable with what can be sensed or perceived, we arrive at the Axiom of Archimedes, in the form in which Leibniz applied it to differentials:

Archimedean Axiom:

Any difference between two terms which is smaller than any that can be given, and thus imperceptible in principle, is null.

Thus, in the context in which he introduces to his Herculean argument, Leibniz writes,

> “... this puts an end to all inquiry about the infinitely small, which cannot be perceived.” [1678-79?; A VI iv 1637; LOC 260]

Acknowledgements

[to be added in the proofs].

References

Arthur R. T. W. (2001a). “Leibniz on Infinite Number, Infinite Wholes and the Whole World: A Reply to Gregory Brown”, *Leibniz Review*, **11**, 103-116.

Arthur R. T. W. (2001b). “Leibniz and Cantor on the Actual Infinite,” pp. 41-46 in *Nihil sine ratione*, Vol. 1, ed. Hans Poser, Berlin: Gottfried-Willhelm-Leibniz-Gesellschaft, 2001.

Arthur R. T. W. (2007). “Actual Infinitesimals in Leibniz’s Early Thought”, forthcoming in a special issue of *Studia Leibnitiana*, ed. Mark Kulstad and Mogens Laerke.

Arthur R. T. W. (2008a). “Leery Bedfellows: Newton and Leibniz on the Status of Infinitesimals,” forthcoming in *Infinitesimals in Leibniz’s Time*, ed. Ursula Goldenbaum and Douglas Jesseph, 2008.

Arthur R. T. W. (2008b). “‘A complete denial of the continuous?’ Leibniz’s philosophy of the continuum,” forthcoming in a special edition of *Synthese* devoted to the Mathematics and Philosophy of the Continuum, 2008.

Arthur R. T. W. (2008c). “Leibniz's Syncategorematic Infinitesimals, Smooth Infinitesimal Analysis, and Newton's Proposition 6”, forthcoming in *Infinitesimals*, ed. William Harper and Wayne C. Myrvold.

Brown, Gregory (2000). “Leibniz on Wholes, Unities and Infinite Number,” *Leibniz Review* **10**, 21-51.

Bos, H. J. M. (1974-75). “Differentials, Higher-Order Differential and the Derivative in the Leibnizian Calculus”, *Archive for the History of the Exact sciences*, **14**, 1-90.

Cantor, Georg (1932). *Gesammelte Abhandlungen*, ed. Ernst Zermelo (Berlin; repr. Hildesheim: Georg Olms, 1962).

Deleuze, Gilles (1994). *Difference and Repetition*. Columbia University Press (English transl. by Paul Patton of *Différence et Repetition* (Presses Universitaires de France, 1968).

Earman, John (1975). “Infinities, Infinitesimals, and Indivisibles: The Leibnizian Labyrinth”. *Sudia Leibnitiana* **7**/2, 236-251.

Guicciardini, Niccolò (2002). “Analysis and synthesis in Newton's mathematical work.” Pp. 308-328 in *The Cambridge Companion to Newton*. Eds. I. Bernard Cohen and George E. Smith. Cambridge: Cambridge University Press.

Guicciardini, Niccolò (2003). “Conceptualism and contextualism in the recent historiography of Newton’s *Principia*,” *Historia Mathematica* 30, 407-431.

Ishiguro, Hidé (1990). *Leibniz’s Philosophy of Logic and Language* (2nd ed.) Cambridge: Cambridge University Press.

Knobloch, Eberhard (2002). “Leibniz’s Rigorous Foundation of Infinitesimal Geometry by means of Riemannian Sums.” *Synthese* 133: 59-73.

Leibniz, G. W. (1675-6/1993). *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*. Ed. and commentary by Eberhard Knobloch. Göttingen: Vandenhoek & Ruprecht.

Leibniz, G. W. (1684). “*Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singulare pro illis calculi genus*.” *Acta Eruditorum*, October 467-473. GM v 220-226.

Leibniz, G. W. (1693). Letter to Foucher, January 1692; *Journal des Sçavans*, March 16, 1693, G.i.416.

Leibniz, G. W. (1701). *“Cum prodiisset atque increbuisset Analysis mea infinitesimalis …”* in Leibniz (1846), pp. 39-50.

Leibniz, G. W. (1846). *Historia et Origo calculi differentialis a G. G. Leibnitzio conscripta* (ed. C. I. Gerhardt). Hannover.

Leibniz, G. W. (2001). *The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672-1686*. Ed., sel. & transl. R. T. W. Arthur. New Haven: Yale University Press.

Leibniz, G. W. (2004). *Quadrature arithmétique du cercle, de l’ellipse et de l’hyperbole*. Ed. Marc Parmentier. Paris: J. Vrin.

Levey, Samuel (2008). “Archimedes, Infinitesimals and the Law of Continuity: On Leibniz’s Fictionalism,” forthcoming in *Infinitesimals in Leibniz’s Time*, ed. Ursula Goldenbaum and Douglas Jesseph, 2008.

Rescher, Nicholas (1967). *The Philosophy of Leibniz*. Englewood Cliffs, NJ: Prentice Hall.